2-LOCAL AUTOMORPHISMS OF OPERATOR ALGEBRAS

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Abstract. A not necessarily continuous, linear or multiplicative function θ from an algebra $\mathcal{A}$ into itself is called a 2-local automorphism if θ agrees with an automorphism of $\mathcal{A}$ at each pair of points in $\mathcal{A}$. In this paper, we study when a 2-local automorphism of a C*-algebra, or a standard operator algebra on a locally convex space, is an automorphism. In particular, if X is a Fréchet space with a Schauder basis and $\mathcal{A}$ contains all locally compact operators on X, every 2-local automorphism θ on $\mathcal{A}$ is an automorphism.

1. Introduction

Let $\mathcal{A}$ be an algebra and θ be a function from $\mathcal{A}$ into $\mathcal{A}$. We call θ an *automorphism* if θ is bijective, linear, and multiplicative. We call θ a 2-*local automorphism* if θ agrees at each pair of points a, b in $\mathcal{A}$ with an automorphism $\theta_{a,b}$ of $\mathcal{A}$, i.e., $\theta(a) = \theta_{a,b}(a)$ and $\theta(b) = \theta_{a,b}(b)$. Note that $\theta_{a,b}$ may depend on a, b.

In [16], Šemrl proves that if H is a separable real or complex Hilbert space then every 2-local automorphism of the algebra $\mathcal{L}(H)$ of all continuous linear operators of H is an automorphism. In [14], Molnár extends this to the following. Let E be a Banach space with a Schauder basis, and let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(E)$ containing all compact operators. Then every 2-local automorphism of $\mathcal{A}$ is an automorphism. Note that in both versions, linearity, surjectivity and continuity are part of the conclusion.

Date: August 31, 2005; a revision submitted to JMAA.

2000 *Mathematics Subject Classification.* 46L40, 47B49, 47L10.

Key words and phrases. local automorphisms, operator algebras, Fréchet spaces.

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We are going to extend these results to the case of standard operator algebras on general locally convex spaces.

Recall that a *standard operator algebra* on a locally convex space X contains the algebra $\mathcal{F}(X)$ of all continuous finite rank operators. We show, without assuming linearity, surjectivity or continuity, that every 2-local automorphism θ of $\mathcal{F}(X)$ is an algebra homomorphism. In case X is a Fréchet space with a Schauder basis and $\mathcal{A}$ contains all locally compact operators, we can conclude that every 2-local automorphism θ on $\mathcal{A}$ is an automorphism. This extends recent results of Šemrl [16] and Molnár [14]. On the other hand, a 2-local automorphism θ of a standard operator algebra $\mathcal{A}$ on a locally convex space X is an algebra homomorphism provided that the range of θ contains $\mathcal{F}(X)$, or θ is continuous in the weak operator topology.

We also study the question when a 2-local automorphism of a C*-algebra is an automorphism.

We are grateful to Matej Brešar and Peter Šemrl for many helpful suggestions. Special thanks are due to Lajos Molnár for reading through a preliminary version of this paper and providing useful advice. We also deeply appreciate the valuable comments from the Referee.

2. 2-local automorphisms of standard operator algebras on locally convex spaces

In this paper, the underlying field can be either the real or the complex. We state some elementary properties of 2-local automorphisms without proofs.

Lemma 2.1. *Let $\theta : \mathcal{A} \to \mathcal{A}$ be a 2-local automorphism of an algebra $\mathcal{A}$.*

(1) θ is homogeneous, that is, $\theta(\lambda a) = \lambda\theta(a)$ for all a in $\mathcal{A}$ and scalar λ.
(2) θ preserves zero products, that is, $ab = 0$ in $\mathcal{A}$ if and only if $\theta(a)\theta(b) = 0$.
(3) θ preserves polynomials, that is, $\theta(p(a)) = p(\theta(a))$ for all polynomials p.
(4) θ is a Jordan homomorphism in case θ is linear.

The following is more or less well known, and the proof is rather straightforward and thus omitted. Note that unless θ below is zero on $\mathcal{F}(X)$, those u and g stated in the statement exist. Here, $z \otimes g$ with an z in X and a g in X' defines the rank at most one operator $(z \otimes g)(x) = g(x)z, \forall x \in X$.

Lemma 2.2. *Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be standard operator algebras on locally convex spaces X and Y, respectively. Let $\theta : \mathcal{A}(X) \to \mathcal{A}(Y)$ be an algebra homomorphism. Suppose $u \in Y$ and $g \in X'$ exist such that $\theta(z \otimes g)u \neq 0$ for some z in X. Define a linear map $T : X \to Y$ by*

$$Tx = \theta(x \otimes g)u, \quad \text{for all } x \text{ in } X.$$

Then T is injective and

$$\theta(A)T = TA, \quad \text{for all } A \text{ in } \mathcal{A}.$$

T is onto in case the range of θ contains $\mathcal{F}(Y)$.

Theorem 2.3. *Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be standard operator algebras on locally convex spaces X and Y, respectively. Let θ be an algebra isomorphism from $\mathcal{A}(X)$ onto $\mathcal{A}(Y)$. Then there is a linear $\sigma(X, X')$–$\sigma(Y, Y')$ homeomorphism T from X onto Y such that*

$$\theta(A) = TAT^{-1}, \quad \text{for all } A \text{ in } \mathcal{A}(X).$$

In case X and Y are Mackey spaces, T is a homeomorphism in the original topologies of X and Y.

Proof. There is a bijective linear map T satisfying $\theta(A)T = TA$ by Lemma 2.2. To see the weak continuity of T, let $\{x_\lambda\}_\lambda$ be a weakly null net in X. Note that the algebra isomorphism θ sends exactly minimal idempotents to minimal idempotents, that is, sends exactly rank one idempotents $x \otimes f$ to rank one idempotents $y \otimes g$ in this case. Here, $f(x) = g(y) = 1$. Observe

$$\begin{aligned} g(Tx_\lambda)y &= (y \otimes g)(Tx_\lambda) = \theta(x \otimes f)Tx_\lambda \\ &= T(x \otimes f)(x_\lambda) = f(x_\lambda)Tx \longrightarrow 0. \end{aligned}$$

We thus see that $g(Tx_\lambda) \to 0$ for all g in Y'. Similarly, T^{-1} is also weakly continuous. Since weakly continuous linear maps are continuous in Mackey spaces (see, e.g., [15, p. 158]), the other assertion follows. □

Remark 2.4. When X, Y are Banach spaces, $\mathcal{A}(X) = \mathcal{L}(X)$, and $\mathcal{A}(Y) = \mathcal{L}(Y)$, Theorem 2.3 is due to Eidelheit [5]. With a different approach, Mackey [13] shows this when X, Y are normed linear spaces. In [17], Vukman proves this when X and Y are bornological locally convex spaces. In an other direction, Šemrl [16] extends this to arbitrary standard operator algebras on normed spaces. Since normed spaces and bornological locally convex spaces are Mackey, Theorem 2.3 generalizes all above. On the other hand, it is possible to extend Theorem 2.3 to ring isomorphisms as in [13]. However, we do not need this generality in this paper.

Some arguments in the proof of following results in this section are borrowed from Šemrl [16] and Molnár [14]. Indeed, they should know of some statements, too. However, we usually provide here shorter and more direct approaches.

Theorem 2.5. *Every 2-local automorphism θ of the algebra $\mathcal{F}(X)$ of continuous finite rank linear operators on a locally convex space X is an algebra homomorphism.*

Proof. First observe that θ preserves the trace of the product of any two elements a, b in $\mathcal{F}(X)$. Indeed,

$$\operatorname{tr}\theta(a)\theta(b) = \operatorname{tr}\theta_{a,b}(a)\theta_{a,b}(b) = \operatorname{tr}\theta_{a,b}(ab) = \operatorname{tr} ab, \tag{2.1}$$

where $\theta_{a,b}$ is the automorphism of $\mathcal{F}(X)$ agreeing with θ at both a and b.

Suppose $\dim X = n < \infty$, and $\{E_{ij} : 1 \leq i, j \leq n\}$ is a set of matrix units. Let $\sum_{i,j} \lambda_{ij}\theta(E_{ij}) = 0$. Multiplying both sides by $\theta(E_{rs})$ and making use of (2.1), we see that $\lambda_{sr} = 0$ for every pair of r and s. In other words, $\{\theta(E_{ij}) : 1 \leq i, j \leq n\}$ is linearly independent, and thus spans $\mathcal{L}(X) \cong M_n$. Again from (2.1), one has

$$\operatorname{tr}\theta(a+b)\theta(c) = \operatorname{tr}(a+b)c = \operatorname{tr} ac + \operatorname{tr} bc = \operatorname{tr}(\theta(a) + \theta(b))\theta(c),$$

or

$$\operatorname{tr}(\theta(a+b) - \theta(a) - \theta(b))\theta(c) = 0, \tag{2.2}$$

for all a, b, c in $\mathcal{F}(X)$. While c runs through all matrix units E_{ij}, one concludes that $\theta(a + b) = \theta(a) + \theta(b)$, and thus θ is a surjective linear Jordan isomorphism from $\mathcal{F}(X) \cong M_n$ onto $\mathcal{F}(X) \cong M_n$. Thus it is either multiplicative or anti-multiplicative (see, e.g., [7]). But the zero product preserving property of θ rules out the second possibility, and thus θ is an automorphism.

In general, let p be an idempotent in $\mathcal{F}(X)$ of rank n. It follows from the local property that $\tilde{p} = \theta(p)$ is also an idempotent of rank n. Denote

$$p\mathcal{F}(X)p = \{a \in \mathcal{F}(X) : a = pap\},$$
$$\tilde{p}\mathcal{F}(X)\tilde{p} = \{b \in \mathcal{F}(X) : b = \tilde{p}b\tilde{p}\}.$$

It is clear that θ gives rise to a 2-local automorphism from $p\mathcal{F}(X)p \cong M_n$ onto $\tilde{p}\mathcal{F}(X)\tilde{p} \cong M_n$. By above discussion, θ is linear and multiplicative on $p\mathcal{F}(X)p$.

Finally, let $a, b \in \mathcal{F}(X)$. Let p be the projection in $\mathcal{F}(X)$ onto a finite dimensional subspace of X containing all the initial and range spaces of a and b, respectively. Therefore, $a = pap$, $b = pbp$, and $a, b \in p\mathcal{F}(X)p$. Consequently, $\theta(a+b) = \theta(a)+\theta(b)$ and $\theta(ab) = \theta(a)\theta(b)$. This implies that θ is an algebra homomorphism of $\mathcal{F}(X)$. □

Corollary 2.6. *Let θ be a 2-local automorphism of a standard operator algebra $\mathcal{A}$ on a locally convex space X. If θ is continuous with respect to the weak operator topology, then θ is an algebra homomorphism.*

Proof. Note that every automorphism of a standard operator algebra sends exactly finite rank operators to finite rank operators. Thus θ induces a 2-local automorphism, and thus an algebra homomorphism, of $\mathcal{F}(X)$ by Theorem 2.5. Since linear sums and products are separately continuous in the weak operator topology, we see that θ also preserves linear sums and products of elements in $\mathcal{A}$. $\square$

Theorem 2.7. *Let θ be a 2-local automorphism of a standard operator algebra $\mathcal{A}$ on a locally convex space X such that its range contains all finite rank operators. Then θ is an algebra homomorphism. More precisely, there is a linear $\sigma(X,X')-\sigma(X,X')$ homeomorphism U on X such that*

$$\theta(T) = UTU^{-1}, \quad \textit{for all } T \textit{ in } \mathcal{A}.$$

In case X is a Mackey space, U is an invertible element in $\mathcal{L}(X)$.

Proof. Note that θ induces a surjective 2-local automorphism, and thus an algebra isomorphism, of $\mathcal{F}(X)$ by Theorem 2.5. It follows from Theorems 2.3 and 2.5 that there exists a linear $\sigma(X,X')-\sigma(X,X')$ homeomorphism U on X such that $\theta(S) = USU^{-1}$ for all S in $\mathcal{F}(X)$. If X is a Mackey space, U is an invertible element in $\mathcal{L}(X)$. In general, let $T \in \mathcal{A}$, $x \in X$ and $f \in X'$. Observe

$$\begin{aligned} f(Tx) &= \operatorname{tr} Tx \otimes f = \operatorname{tr} T \cdot x \otimes f \\ &= \operatorname{tr} \theta(T)\theta(x \otimes f) = \operatorname{tr} \theta(T)(Ux \otimes f \circ U^{-1}) \\ &= f(U^{-1}\theta(T)Ux). \end{aligned}$$

Consequently, $\theta(T) = UTU^{-1}$ for all T in $\mathcal{A}$. $\square$

The following example shows that the condition the range of θ containing $\mathcal{F}(X)$ is indispensable in last theorem. Moreover, we also see that a bounded linear local automorphism of a dense subalgebra of a C*-algebra might not be extended to such one of the whole. Here, we call θ an n-local automorphism of an algebra $\mathcal{A}$ if for any arbitrary n elements $a_1, \ldots, a_n$ of $\mathcal{A}$ there is an automorphism $\theta_{a_1,\ldots,a_n}$ of $\mathcal{A}$ such that

$\theta(a_i) = \theta_{a_1,\dots,a_n}(a_i)$ for $i = 1, \dots, n$. Usually, a 1-local automorphism is called a local automorphism.

Example 2.8. Let R and L be the unilateral shift and the backward unilateral shift on an infinite dimensional separable Hilbert space H with respect to an orthonormal basis $\{e_n : n = 1, 2, \dots\}$, respectively. Let θ be defined by

$$\theta(A) = RAL, \quad \text{for all } A \text{ in } \mathcal{F}(H).$$

It is easy to see that θ is a non-surjective linear n-local automorphism of $\mathcal{F}(H)$ for $n = 1, 2, \dots$. Indeed, θ is an isometric algebra homomorphism of $\mathcal{F}(H)$. However, there is no invertible U in $\mathcal{L}(H)$ such that $\theta(A) = UAU^{-1}$ for all A in $\mathcal{F}(H)$.

On the other hand, θ cannot be extended to a bounded linear local automorphism of the algebra $\mathcal{K}(H)$ of all compact operators on H, which is the norm closure of $\mathcal{F}(H)$. For else, we had to have

$$\theta(\sum_{n=1}^{\infty} \frac{1}{n}\, e_n \otimes e_n) = \sum_{n=1}^{\infty} \frac{1}{n}\, e_{n+1} \otimes e_{n+1}.$$

However, $\sum_{n=1}^{\infty} \frac{1}{n}\, e_n \otimes e_n$ is injective and thus not a left zero divisor of $\mathcal{K}(H)$, while the one at the right hand side is. This conflicts with the local property of θ.

3. 2-local automorphisms of operator algebras on a Fréchet space with a Schauder basis

In case X is a Fréchet space with a Schauder basis, we can drop both the surjectivity and continuity assumption of θ in previous results. Recall that a sequence $\{x_n\}_n$ in a topological vector space V is called a *topological basis* if every x in V determines a unique scalar sequence $\{\xi_n\}_n$ such that $x = \sum_n \xi_n x_n$; this is either a finite sum or a convergent series in the topology of V. By setting $f_n(x) = \xi_n$, we obtain a linear functional on V, called the *n-th coefficient functional* corresponding to $\{x_n\}_n$. A *Schauder basis* $\{x_n\}_n$ is a topological basis with continuous coefficient functionals

$\{f_n\}_n$. In the following, X always denotes a Fréchet space with a Schauder basis $\{e_1, e_2, \ldots\}$, and $\{f_1, f_2, \ldots\}$ is the dual basis of X'.

Lemma 3.1. *For each continuous seminorm q of X, there is a continuous seminorm p of X such that*

$$|f_n(x)|q(e_n) \leq p(x), \quad \textit{for all } x \textit{ in } X \textit{ and } n = 1, 2, \ldots.$$

Proof. For each x in X, we can write

$$x = \sum_n f_n(x)e_n.$$

In particular, $f_n(x)e_n$ converges to zero in X. Let

$$X_m = \{x \in X : |f_n(x)|q(e_n) \leq m, \ \forall n = 1, 2, \ldots\}, \quad m = 1, 2, \cdots.$$

By the Baire Category Theorem, the fact that $X = \bigcup_m X_m$ ensures at least one of them has nonempty interior. Thus there is a continuous seminorm p' of X such that

$$p'(x) \leq 1 \Rightarrow |f_n(x)|q(e_n) \leq m, \quad \text{for } n = 1, 2, \cdots.$$

Consequently,

$$|f_n(x)|q(e_n) \leq mp'(x), \quad \text{for all } x \text{ in } X \text{ and } n = 1, 2, \ldots.$$

Set $p = mp'$. □

We remark that a topological basis of a locally convex space is uniform in the sense of Lemma 3.1 above if and only if it is a Schauder basis. To make up a proof for the converse, consult, e.g., [8, Theorem 14.3.6].

Recall that a *locally compact* (resp. *compact*) linear operator between locally convex spaces is the one sending bounded sets (resp. a neighborhood of zero) to relatively compact sets. It is plain that a compact operator is locally compact. The converse might not hold unless the underlying space is a normed space.

Lemma 3.2. *The formula*

$$Sx = \sum_n \lambda_n f_n(x) e_n$$

defines a locally compact linear operator S on X for every scalar sequence (λ_n) in ℓ_1.

Proof. We first show that $\sum_n \lambda_n f_n(x) e_n$ converges in X. Indeed, for each continuous seminorm q of X, by Lemma 3.1 we have a continuous seminorm p of X such that

$$\sum_n |\lambda_n||f_n(x)|q(e_n) \leq \sum_n |\lambda_n| p(x) = p(x) \sum_n |\lambda_n| < +\infty.$$

Hence $\sum_n \lambda_n f_n(x) e_n$ is absolutely convergent in the Fréchet space X, and thus converges for all x in X. The linearity of S is obvious. To see S is continuous, we suppose $y_m \to 0$ and $Sy_m \to z$ in X. Note

$$Sy_m = \sum_n \lambda_n f_n(y_m) e_n.$$

Applying f_n to both sides, we get

$$f_n(Sy_m) = \lambda_n f_n(y_m), \quad m, n = 1, 2, \ldots.$$

Letting n fixed but m to infinity, we have

$$f_n(z) = \lim_m \lambda_n f_n(y_m) = 0.$$

Consequently,

$$z = \sum_n f_n(z) e_n = 0.$$

The Closed Graph Theorem establishes the continuity of S.

Let B be a bounded subset of X. We claim that SB is relatively compact in X. Let y_m be in B for $m = 1, 2, \ldots$. Observe that

$$Sy_m = \sum_n \lambda_n f_n(y_m) e_n, \quad \text{for all } m = 1, 2, \ldots.$$

Since $\{f_n(y_m) : m = 1, 2, \ldots\}$ is a bounded set of scalars, it is relatively compact. By a diagonal argument, we have a subsequence $\{y_{m_k}\}_k$ such that all $f_n(y_{m_k})$ converges to some scalars β_n as $k \to \infty$. By Lemma 3.1, one sees that $\sum_n \lambda_n \beta_n e_n$ converges in X and the sum is the limit of Sy_{m_k}. Thus S is locally compact. □

Theorem 3.3. *Let X be a real or complex Fréchet space with a Schauder basis, and $\mathcal{A}$ a subalgebra of $\mathcal{L}(X)$ containing all locally compact operators on X. Then every 2-local automorphism θ of $\mathcal{A}$ is an automorphism. More precisely, there is an invertible continuous linear operator T on X such that*

$$\theta(A) = TAT^{-1}, \quad \textit{for all } A \ \textit{ in } \mathcal{A}.$$

Proof. We follow the plan of Molnár [14], which he uses to deal with the case X is a Banach space with a Schauder basis. However, some extra efforts are introduced due to the new generality. The case that X is of finite dimension is established. Assume now X is infinite dimensional. Let $P_n = e_n \otimes f_n$, where $\{e_n : n = 1, 2, \ldots\}$ is a Schauder basis of X with continuous coefficient functionals f_n's. Set $\lambda_n = (\frac{1}{3})^n$ for $n = 1, 2, \ldots$. By Lemma 3.2, the sum $\sum_n \lambda_n P_n$ converges strongly to a locally compact operator. By the local property of θ, composing θ with an automorphism of $\mathcal{A}$ if necessary, we can assume for the particular operator $\sum_n \lambda_n P_n$ that

$$\theta(\sum_n \lambda_n P_n) = \sum_n \lambda_n P_n.$$

Claim 1. $\theta(P_n) = P_n$ for $n = 1, 2, \ldots$.

Let n_0 in $\mathbb{N}$ be arbitrary. By the local property of θ and Theorem 2.3, we have an invertible continuous linear operator U in $\mathcal{L}(X)$ such that

$$\theta(\sum_n \lambda_n P_n) = U(\sum_n \lambda_n P_n)U^{-1} \quad \text{and} \quad \theta(P_{n_0}) = UP_{n_0}U^{-1}.$$

Let $Q_n = UP_nU^{-1}$ for $n = 1, 2, \ldots$. Divide both sides of the equality

$$\sum_n \lambda_n P_n = \sum_n \lambda_n Q_n$$

by λ_1 and get

$$P_1 + \frac{\sum_{n=2}^{\infty} \lambda_n P_n}{\lambda_1} = Q_1 + \frac{\sum_{n=2}^{\infty} \lambda_n Q_n}{\lambda_1}.$$

Taking the kth powers, we have

$$P_1 + \frac{\sum_{n=2}^{\infty} \lambda_n^k P_n}{\lambda_1^k} = Q_1 + \frac{\sum_{n=2}^{\infty} \lambda_n^k Q_n}{\lambda_1^k},$$

since all $\{P_n\}_n$'s (resp. $\{Q_n\}_n$'s) are disjoint rank one idempotents. By Lemma 3.1, for each continuous seminorm q of X, there exists a continuous seminorm p of X such that

$$q(P_n(x)) = q(f_n(x)e_n) \leq p(x) \quad \text{for all } x \text{ in } X, \quad \text{for all } n = 1, 2, \ldots.$$

So,

$$q(\frac{\sum_{n=2}^{\infty} \lambda_n^k P_n x}{\lambda_1^k}) \leq \frac{\sum_{n=2}^{\infty} \lambda_n^k q(P_n x)}{\lambda_1^k} \leq \frac{(\sum_{n=2}^{\infty} \lambda_n)^k}{\lambda_1^k} p(x), \quad \text{for all } x \text{ in } X.$$

We can also obtain a similar result for $\frac{\sum_{n=2}^{\infty} \lambda_n^k Q_n x}{\lambda_1^k}$. Letting k tend to infinity, as $\sum_{n\geq 2} \lambda_n < \lambda_1$, we conclude that $Q_1 = P_1$, and therefore,

$$\sum_{n=2}^{\infty} \lambda_n P_n = \sum_{n=2}^{\infty} \lambda_n Q_n.$$

One can proceed in the same way to show that $Q_n = P_n$ holds for every $n = 1, 2, \ldots$. In particular, we have $\theta(P_{n_0}) = Q_{n_0} = P_{n_0}$. But n_0 is arbitrary, we conclude that

$$\theta(P_n) = P_n \quad \text{for all } n = 1, 2, \ldots.$$

By Lemma 2.2 and Theorem 2.5, there is an injective linear operator $T : X \to X$ such that

$$TA = \theta(A)T, \quad \text{for all } A \text{ in } \mathcal{F}(X). \tag{3.1}$$

Claim 2. T is continuous and onto.

Let $\{x_n\}_n$ be a sequence in X and y in X be such that $x_n \to 0$ and $Tx_n \to y$. We show that $y = 0$. For every A in $\mathcal{F}(X)$, TA is continuous, and thus we have

$TAx_n \to 0$. By (3.1), it implies $\theta(A)Tx_n \to 0$. Because $\theta(A)$ is continuous and $Tx_n \to y$, we get $\theta(A)y = 0$. By Claim 1, we have

$$y = \sum_n f_n(y)e_n = \sum_n P_n(y) = \sum_n \theta(P_n)y = 0.$$

The continuity of T follows by the Closed Graph Theorem.

On the other hand, $TP_n = P_nT$ by (3.1). Thus

$$Te_n = \alpha_n e_n, \quad \text{for some nonzero scalars } \alpha_n,\ n = 1, 2, \ldots. \tag{3.2}$$

This implies T has dense range. For every x in X and f in X', we have by the local property of θ and by (3.1) again that

$$Tx \otimes f = \theta(x \otimes f)T = Wx \otimes T'(W^{-1})'f,$$

where W is an invertible linear operator in $\mathcal{L}(X)$ as in Theorem 2.3. Consequently, f belongs to the range of T' for all f in X'. Now the fact T' is onto ensures that T has closed range (see, e.g., [15, Page 160]), and thus T has a continuous inverse by the Open Mapping Theorem.

At this point, we have shown that $\theta(A) = TAT^{-1}$ for all A in $\mathcal{F}(X)$. As in the proof of Theorem 2.7, we can show that

$$\theta(A) = TAT^{-1}, \quad \text{for all } A \text{ in } \mathcal{A}.$$

To complete the proof it remains to show that θ is surjective. Let $B \in \mathcal{A}$. By the local property of θ and Theorem 2.3, there is an invertible continuous linear operator V on X such that

$$\sum_n \lambda_n P_n = \theta(\sum_n \lambda_n P_n) = V(\sum_n \lambda_n P_n)V^{-1} = \sum_n \lambda_n V P_n V^{-1},$$

and

$$\theta(B) = VBV^{-1}.$$

Since $A \mapsto VAV^{-1}$ is an automorphism of $\mathcal{A}$, we have $V\mathcal{A}V^{-1} = \mathcal{A}$ or $V^{-1}\mathcal{A}V = \mathcal{A}$. By repeating some arguments above, we have

$$VP_nV^{-1} = P_n \quad \text{or} \quad VP_n = P_nV, \quad \text{for all } n = 1, 2, \cdots .$$

It follows that

$$Ve_n = \beta_n e_n \quad \text{for some nonzero scalars } \beta_n,\ n = 1, 2, \ldots .$$

Together with (3.2), we have $VTe_n = TVe_n$ for $n = 1, 2, \ldots$, and thus $VT = TV$ since $\{e_n : n = 1, 2, \ldots\}$ is a Schauder basis of X. Therefore,

$$T^{-1}BT = V^{-1}T^{-1}VBV^{-1}TV = V^{-1}T^{-1}\theta(B)TV = V^{-1}BV \in \mathcal{A}.$$

Consequently,

$$B = T(T^{-1}BT)T^{-1} = \theta(T^{-1}BT) \in \theta(\mathcal{A}).$$

This establishes the surjectivity of θ, and completes the proof. □

4. 2-local automorphisms of C*-algebras

Since automorphisms of abelian C*-algebras $C_0(X)$ are linear isometries, every 2-local automorphism θ of $C_0(X)$ preserves distance as

$$\|\theta(a) - \theta(b)\| = \|\theta_{a,b}(a - b)\| = \|a - b\|, \quad \text{for all } a, b \text{ in } C_0(X).$$

It follows from the Mazur-Ulam Theorem that surjective 2-local automorphisms of abelian C*-algebras are real, and thus complex by Lemma 2.1, linear isometry. It then follows from the Banach-Stone Theorem that they are automorphisms, and carry the form $\theta(f) = f \circ \phi$ for some homeomorphism ϕ of X. Indeed, a little more can be said with the same argument. One can find a generalized version of the Banach-Stone theorem in [4] to finish a proof of the following

Proposition 4.1. *Every surjective 2-local automorphism of a regular uniform real or complex function algebra is an automorphism.*

The following theorem is a special case of a recent result of Györy. In fact, Györy proved a seemingly stronger result in [6] for the so-called 2-local isometries, that is the one agreeing with a surjective linear isometry at each pair of points. A new and short proof working only for 2-local automorphisms is provided below for completeness. Here the underlying field is the complex. Interested readers are referred to Jarosz and Rao [9] for more discussions on local isometries.

Proposition 4.2. *Every 2-local automorphism θ of $C_0(X)$ is an algebra homomorphism. In case the locally compact Hausdorff space X is first countable, θ is an automorphism.*

Proof. First recall the following result of Kowalski and Słodkowski [11]. Let $\mathcal{A}$ be a complex Banach algebra (not necessarily commutative nor unital). Let $f : A \to \mathbb{C}$ satisfy that $f(0) = 0$ and

$$f(a) - f(b) \in \sigma(a - b), \quad \text{for all } a, b \text{ in } \mathcal{A}.$$

Then f is linear and multiplicative. Here, $\sigma(a - b)$ denotes the spectrum of $a - b$.

Let $z \in X$ and set

$$f = \delta_z \circ \theta : C_0(X) \to \mathbb{C}.$$

Then

$$\begin{aligned} f(a) - f(b) &= \delta_z \circ \theta(a) - \delta_z \circ \theta(b) = \delta_z \circ \theta_{a,b}(a) - \delta_z \circ \theta_{a,b}(b) \\ &= \theta_{a,b}(a - b)(z) \in \sigma(a - b), \quad \text{for all } a, b \text{ in } C_0(X). \end{aligned}$$

Consequently, $\delta_z \circ \theta$ is a nonzero multiplicative linear functional of $C_0(X)$. Hence, there is a point $\phi(z)$ in X such that $\delta_z \circ \theta = \delta_{\phi(z)}$. In other words,

$$\theta(a)(z) = a(\phi(z)), \quad \text{for all } a \text{ in } C_0(X).$$

Therefore, θ is an isometric algebra homomorphism. It follows from [10, Theorem 1] that the map $\phi : X \to X$ is continuous, open and onto.

Assume now that X is first countable. We show that ϕ is one-to-one. Suppose $\phi(x) = \phi(y) = z$. Let a be a continuous function in $C_0(X)$ peak at z; namely, $0 \leq a \leq 1$ and a assumes value 1 exactly at the point z. Since $\theta(a) = a \circ \phi_a$ for some homeomorphism ϕ_a of X, we see that $\theta(a) = a \circ \phi$ peaks at exactly one point. This forces $x = y$. Therefore, ϕ is a homeomorphism and θ is an automorphism. □

We note that not every 2-local automorphism of $C_0(X)$ is surjective. In fact, one can see that the map θ in [12, Example 3.3] is a non-surjective linear n-local automorphism of $C[0, \beta]$, where $n = 1, 2, \ldots, \omega$, and ω (resp. β) is the first countably infinite (resp. uncountably) ordinal number. Here, $[0, \beta]$ is not first countable as β is not a G_δ-point. On the other hand, there is an example of Crist [3] for a 2-local automorphism of a subalgebra of the algebra M_3 of 3×3 matrices, which is not linear. But for linear 2-local automorphisms of C*-algebras, we have a positive result. Let us recall the following

Theorem 4.3 ([2])**.** *Let θ be a surjective bounded linear map from a C*-algebra $\mathcal{A}$ onto a C*-algebra $\mathcal{B}$ preserving zero products. Then $\theta(a) = \theta(1)\varphi(a) = \varphi(a)\theta(1)$ for all a in $\mathcal{A}$, where φ is an algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*

Theorem 4.4. *Let θ be a linear 2-local automorphism of a C*-algebra. If the range of θ is a C*-algebra then θ is an algebra homomorphism.*

Proof. Note that θ is bounded since it preserves spectra (see, e.g., [1]). The assertion is now a consequence of Lemma 2.1 and Theorem 4.3. □

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