LOCAL AUTOMORPHISMS OF OPERATOR ALGEBRAS

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Abstract. A not necessarily continuous, linear or multiplicative function θ from an algebra $\mathcal{A}$ into itself is called a local automorphism if θ agrees with an automorphism of $\mathcal{A}$ at each point in $\mathcal{A}$. In this paper, we study the question when a local automorphism of a C*-algebra, or a W*-algebra, is an automorphism.

1. Introduction

Let $\mathcal{A}$ be an algebra and θ be a function from $\mathcal{A}$ into $\mathcal{A}$. We call θ an *automorphism* if θ is bijective, linear, and multiplicative. We call θ a *local automorphism* if θ agrees at each point a in $\mathcal{A}$ with an automorphism θ_a of $\mathcal{A}$, i.e., $\theta(a) = \theta_a(a)$. Note that θ_a may depend on a. This notion obviously relates to the properties of preserving invertibility, commutativity, idempotents, square zero elements, and more important, spectra (see, e.g., [13, 7, 27, 31, 9, 10, 11, 29]). The potential applications in mathematical physics is also clear (see, e.g., [25]). In this paper, we will investigate when a local automorphism of an operator algebra is an automorphism.

A local automorphism sends 0 to 0, and 1 to 1 in case $\mathcal{A}$ is unital, but else it can be arbitrary. For example, let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Define an equivalence relation on $B(H)$ by saying that A and B are equivalent if there is a unitary operator U on H such that $A = UBU^*$. Assign to each member in an equivalence class $[A]$ the same unitary $U_{[A]}$, and then define $\theta : B(H) \to B(H)$ by

$$\theta(A) = U_{[A]} A U_{[A]}^*, \quad \text{for all } A \text{ in } B(H).$$

It is easy to see that θ is a bijective local automorphism of $B(H)$ preserving norm. Unless all $U_{[A]}$ are equal, however, θ does not observe any algebraic structure of $B(H)$.

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To get a meaningful theory it seems to be necessary to assume linearity, surjectivity and/or continuity of a local automorphism. Note that injectivity is free whenever linearity presents. On the other hand, local automorphisms are spectrum preserving. It then follows from a result of Aupetit that a surjective linear local automorphism of a semisimple Banach algebra is automatically bounded (see, e.g., [2]). But such linear (and thus continuous and injective) automorphisms can be not surjective (see Example 3.3 below, and see also [24, Example 2.8]).

The notion of local automorphisms is introduced by Larson and Sourour [23]. They showed that every invertible linear local automorphism of a matrix algebra is either an automorphism or an anti-automorphism, and that of $B(H)$ is an automorphism whenever H is an infinite dimensional Hilbert space (see also Brešar and Šemrl [8].)

In this paper, we will see that a surjective linear local automorphism θ of a von Neumann algebra $\mathcal{N}$ is a Jordan isomorphism. In case $\mathcal{N}$ is properly infinite, θ is an automorphism. On the other hand, linear local automorphisms of abelian C*-algebras are always algebra homomorphisms. They are not necessarily surjective, however. A sufficient condition ensuring surjectivity is that the pure state space is first countable, and a counter example is provided when this does not hold.

We do not know too much about linear local automorphisms of non-abelian C*-algebras, except for those with real rank zero. In comparison, there is a similar concept called *local derivations*. In [21], Kadison showed that every bounded linear local derivation of a von Neumann algebra is a derivation, and in [30], Shul'man extended this to the case of C*-algebras. See also similar results of Brešar [6] and Johnson [20].

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2. Local automorphisms of W*-algebras

We first state some properties of a local automorphism without proof.

Lemma 2.1. *Let θ be a local automorphism of an algebra $\mathcal{A}$.*

(1) θ preserves k-potents for $k = 2, 3, \ldots$; more precisely, $a^k = a$ if and only if $\theta(a)^k = \theta(a)$.

(2) θ *preserves* k*-power zero elements; more precisely,* $a^k = 0$ *if and only if* $\theta(a)^k = 0$.

(3) θ *preserves central elements.*

(4) θ *preserves left (resp. right, two-sided) zero divisors.*

(5) θ *preserves zeros of polynomials, and thus algebraic elements.*

(6) If $\mathcal{A}$ *is unital, then* θ *preserves left (resp. right, two-sided) invertibility.*

(7) If $\mathcal{A}$ *is unital, then* θ *preserves left (resp. right, two-sided) spectra.*

(8) If θ *is linear, then we can extend* θ *uniquely to a local automorphism of the unitalization of* $\mathcal{A}$ *by setting* $\theta(1) = 1$.

In [23], Larson and Sourour show that every linear local automorphism of the matrix algebra $M_n(\mathbb{C})$ is either of the form $A \mapsto TAT^{-1}$ or of the form $A \mapsto TA^tT^{-1}$ for some nonsingular matrix T. Indeed, a matrix A and its transpose A^t have the same Jordan form, and thus A and A^t are similar to each other. Therefore, the map $A \mapsto A^t$ is a surjective linear local automorphism, but not an automorphism for $n > 1$.

Recall that a Jordan homomorphism of an algebra is a linear map preserving the Jordan product $a \circ b = ab + ba$. The following result was proved by Brešar and Šemrl [11]. See also [6, 7]. We sketch the proof here for completeness.

Theorem 2.2 (Brešar and Šemrl)**.** *Every bounded linear local automorphism* θ *of a* W^**-algebra* $\mathcal{N}$ *is a Jordan homomorphism.*

Proof. By Lemma 2.1, θ sends idempotent elements to idempotent elements. It follows that θ sends orthogonal idempotents to orthogonal idempotents. By the spectral theory, every self-adjoint element a in $\mathcal{N}$ can be approximated in norm by linear sums of orthogonal projections. More precisely,

$$a = \lim_n \sum_k \lambda_{nk} P_{nk},$$

for some families of finitely many orthogonal projections P_{nk}. By the boundedness of θ, we have

$$\theta(a) = \lim_n \sum_k \lambda_{nk} \theta(P_{nk}).$$

The above observation implies that

$$\theta(a)^2 = \lim_n \sum_k \lambda_{nk}^2 \theta(P_{nk}) = \theta(a^2).$$

Now for all self-adjoint a, b in $\mathcal{N}$, the equality $\theta((a+b)^2) = (\theta(a+b))^2$ gives $\theta(ab+ba) = \theta(a)\theta(b)+\theta(b)\theta(a)$. We see that θ is a Jordan homomorphism by observing the equality $(\theta(a+ib))^2 = (\theta(a)+i\theta(b))^2 = \theta((a+ib)^2)$. □

We provide a refinement of Theorem 2.2 below.

Theorem 2.3. *Suppose the range of a linear local automorphism θ of a W^*-algebra $\mathcal{N}$ is a W^*-algebra. Then θ is automatically bounded, and thus a Jordan homomorphism. If, in addition, $\mathcal{N}$ is properly infinite, then θ is an algebra homomorphism.*

Proof. The first assertion was proved in [15]. Indeed, surjective spectrum preserving linear maps between semisimple Banach algebras are automatically bounded (see, e.g., [2]). By Theorem 2.2, we see that θ is a Jordan homomorphism.

From now on, suppose $\mathcal{N}$ is properly infinite. That is, every nonzero central projection in $\mathcal{N}$ is infinite. By a result of Brešar [5] (see also [1]), there are σ-weakly closed ideals I, J of $\mathcal{N}$ and ideals I', J' of $\theta(\mathcal{N})$ such that $\mathcal{N} = I \oplus J$, $\theta(\mathcal{N}) = I' \oplus J'$, and θ induces an algebra isomorphism from I onto I' and an anti-isomorphism from J onto J'. In particular, $\theta(ab) = \theta(b)\theta(a)$ for all a, b in J.

Suppose J is not zero, for else we are done. Let $1_I, 1_J$ be the orthogonal central projections in $\mathcal{N}$ such that $I = 1_I\mathcal{N}$ and $J = 1_J\mathcal{N}$. Since 1_J is not finite, there is a partial isometry p in J such that $p^*p = 1_J$ but $pp^* < 1_J$. Observe

$$(p^* + 1_I)(p + 1_I) = p^*p + 1_I = 1,$$

$$(p + 1_I)(p^* + 1_I) = pp^* + 1_I < 1.$$

Hence, $p + 1_I$ is not right invertible. It follows from Lemma 2.1 that $\theta(p + 1_I)$ is not right invertible, either. On the other hand,

$$\begin{aligned} 1 = \theta(1) &= \theta((p^* + 1_I)(p + 1_I)) \\ &= \theta(p^*p) + \theta(1_I) = \theta(p)\theta(p^*) + \theta(1_I) \\ &= (\theta(p) + \theta(1_I))(\theta(p^*) + \theta(1_I)). \end{aligned}$$

This says $\theta(p + 1_I)$ is right invertible, a contradiction. $\square$

A linear local automorphism θ of a von Neumann algebra $\mathcal{N}$ sends central projections to central idempotents, indeed projections, as θ also preserves spectra. Let $I = \mathcal{N}p$ be a σ-weakly closed two-sided ideal of $\mathcal{N}$ with p a central projection in $\mathcal{N}$. By Theorem 2.2, θ preserves Jordan products, and thus

$$\theta(ap) = (\theta(a)\theta(p) + \theta(p)\theta(a))/2 = \theta(a)\theta(p), \quad \forall a \in \mathcal{N}.$$

Hence, $\theta(I) = \theta(\mathcal{N})\theta(p)$ is also a σ-weakly closed two-sided ideal of $\mathcal{N}$ if θ is surjective. By a result of Sakai [28, Corollary 4.1.23], every algebra isomorphism between two W*-algebras are of the form $a \mapsto \pi(uau^{-1})$ where π is a σ-weakly bi-continuous $*$-isomorphism and u is an invertible element in the domain. A similar result also holds for algebra anti-isomorphisms. Thus, θ preserves types of ideals, too. In view of Theorem 2.3 and results of Larson and Sourour [23], and Brešar and Šemrl [8], there is just only one case not completely clear to us at this moment, and we make it as a

Problem 2.4. Can a surjective linear local automorphism of a von Neumann algebra of type II_1 be an anti-automorphism?

3. Local automorphisms of C*-algebras

Some of above arguments also apply to linear local automorphisms of C*-algebras of real rank zero. However, another result of Brešar [4] about the structure of Jordan homomorphisms between C*-algebras might be used instead of that in [5] (see also [12]). Note that every self-adjoint element in such an algebra can also be approximated in norm by linear sums of orthogonal idempotents. Recall also that a C*-algebra is *purely infinite* if every hereditary C*-subalgebra is infinite.

Theorem 3.1. *Let θ be a linear local automorphism of a C*-algebra $\mathcal{A}$ of real rank zero. Suppose the range of θ is a C*-algebra. Then θ is a Jordan homomorphism. If, in addition, A is purely infinite, then θ is an automorphism.*

Due to the lack of projections, we do not know whether the above theorem holds or not if the C*-algebra is not of real rank zero. However, the abelian case is completely done. The following result is due to Molnár and Zalar [26]. We sketch a proof here for completeness.

Theorem 3.2 ([26]). *Every complex linear local automorphism θ of an abelian C^*-algebra $\mathcal{A} = C_0(X)$ is an isometric algebra homomorphism. In case X is first countable, θ is an automorphism.*

Proof. Note that every isometric algebra homomorphism (resp. automorphism) of $C_0(X)$ arises from a composition $f \mapsto f \circ \phi$ with a quotient map (resp. homeomorphism) ϕ from X onto X (see, e.g., [16]).

Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X. Setting $\theta(1) = 1$, we can also consider that θ is a linear local automorphism of $C(X_\infty)$. For an f in $C(X_\infty)$, the spectrum of f coincides with its range $\sigma(f) = f(X_\infty)$. In particular, the norm of f equals its spectral radius, and f is invertible exactly when f is non-vanishing on X_∞. By Lemma 2.1, θ preserves both norm and invertibility (i.e. being non-vanishing). By the Gleason-Kahane-Zelazko Theorem [17, 22] (see also [18]), θ is multiplicative, and thus an isometric algebra homomorphism of $C(X_\infty)$. More precisely, $\theta(f) = f \circ \phi$, where the map $\phi : X_\infty \to X_\infty$ is continuous, open and onto. Clearly, ϕ sends exactly ∞ to ∞. Hence, we can also think of ϕ as a quotient map from X onto X, and θ as an isometric algebra homomorphism of $C_0(X)$.

Assume now that X is first countable. We show that ϕ is one-to-one. Suppose $\phi(x) = \phi(y) = z$. Let f be a continuous function in $C_0(X)$ peak at z; namely, $0 \leq f \leq 1$ and f assumes value 1 exactly at the point z. Since $\theta(f) = f \circ \phi_f$ for some homeomorphism ϕ_f of X, the function $\theta(f) = f \circ \phi$ peaks at exactly one point. This forces $x = y$. Therefore, ϕ is a homeomorphism and θ is an automorphism. □

In the following example, we see that a linear local automorphism of $C(X)$ needs not be surjective if X contains a non-G_δ point.

Example 3.3. Let ω and β be the first infinite and the first uncountable ordinal number, respectively. Let $[0, \beta]$ be the compact Hausdorff space consisting of all ordinal numbers x not greater than β and equipped with the topology generated by order intervals. Note that every continuous function f in $C[0, \beta]$ is eventually constant.

More precisely, there is a non-limit ordinal x_f such that $\omega < x_f < \beta$ and $f(x) = f(\beta)$ for all $x \geq x_f$.

Define $\phi : [0, \beta] \to [0, \beta]$ by setting

$$\phi(0) = \beta, \quad \phi(n) = n - 1 \text{ for all } n = 1, 2, \ldots, \quad \text{and } \phi(x) = x \text{ for all } x \geq \omega.$$

Let $\theta : C[0, \beta] \to C[0, \beta]$ be the non-surjective composition operator defined by $\theta(f) = f \circ \phi$. We shall see that θ is an isometric linear local automorphism. Indeed, θ is clearly isometric and linear. For each f in $C[0, \beta]$, let ϕ_f be the homeomorphism of $[0, \beta]$ defined by

$$\phi_f(0) = x_f, \quad \phi_f(n) = n - 1 \text{ for all } n = 1, 2, \ldots, \quad \phi_f(x) = x \text{ for all } \omega \leq x < x_f,$$

$$\phi_f(x) = x + 1 \text{ for all } x_f \leq x < x_f + \omega, \quad \text{and } \phi_f(x) = x \text{ for all } x \geq x_f + \omega.$$

It is plain that $\theta(f) = f \circ \phi = f \circ \phi_f$ for all f in $C[0, \beta]$.

Note that to utilize the Gleason-Kahane-Zelazko Theorem [17, 22] in the proof of Theorem 3.2, the underlying field is assumed to be the complex. We are expecting a new proof for the real case. Here is a partial solution.

Proposition 3.4. *Suppose the underlying field is the real,* $\mathbb{R}$. *Let* X *be a locally compact subset of* $\mathbb{R}$. *Then every linear local automorphism* θ *of* $C_0(X)$ *is an automorphism.*

Proof. It follows from the local property that θ is a linear isometry. By an extension of the Hołsztynski Theorem [19], there is a locally compact subset Y of X and a surjective continuous open map ϕ from Y onto X such that

$$\theta(f)|_Y = f \circ \phi. \tag{3.1}$$

It follows from a similar argument as in the proof of Theorem 3.2 that ϕ is one-to-one, and thus a homeomorphism.

We shall construct a strictly positive function f in $C_0(X)$ with the property that each *level set* $f^{-1}(\lambda) = \{x \in X : f(x) = \lambda\}$ is finite for all $\lambda > 0$. Note that X is the union of all level sets of f. Suppose we have such an f for this moment. By the local property, $\theta(f) = f \circ \phi_f$ is also a function of such kind. For each $\lambda > 0$, suppose

$f^{-1}(\lambda)$ consists of distinct points $x_1, x_2, \ldots, x_n$ in X. Since ϕ is bijective, there are distinct points $y_1, y_2, \ldots, y_n$ in Y with $\phi(y_i) = x_i$ for $i = 1, 2, \ldots, n$. It follows from (3.1) that $f(\phi_f(y_i)) = f(\phi(y_i)) = \lambda$ for $i = 1, 2, \ldots, n$. By counting elements, we see that the points $y_1, y_2, \ldots, y_n$ enumerates all of the λ-level set of $f \circ \phi_f$. In particular, all the level sets of $f \circ \phi_f$ are contained in Y. Consequently, $X = Y$, and thus θ is an automorphism of $C_0(X)$.

Now, we construct such an f in $C_0(X)$. For each x in X, by the local compactness, there are $a < b$ such that $X \cap [a, b]$ is a compact neighborhood of x in X. Let α be the infimum of all such a and β be the supremum of all such b in $\mathbb{R}$. Here, we allow $\alpha = -\infty$ and $\beta = +\infty$. Using this idea, we can write X as a countable disjoint union $X = \cup_n X_n$, where each $X_n = X \cap [\alpha_n, \beta_n]$ for some $\alpha_n < \beta_n$ has the property that $X \cap [a, b]$ is compact in X for all $\alpha_n < a < b < \beta_n$.

Choose an f_n in $C_0(X)$ vanishing outside (α_n, β_n). The behavior of f_n on X_n depends on whether X contains the endpoints α_n, β_n. If X_n does not contain either of α_n, β_n, we assume f_n agrees on X_n with a continuous function which joins the points $(\alpha_n, 0)$, $(\frac{\alpha_n+\beta_n}{2}, 1/n)$ and $(\beta_n, 0)$ in the plane firstly by a strictly increasing curve and then by a strictly decreasing one. In case X_n contains α_n but not β_n, we assume f_n agrees on X_n with a strictly decreasing curve passing through the points $(\alpha_n, 1/n)$ and $(\beta_n, 0)$. A similar construction is applied to the situation that X_n contains β_n but not α_n. If X_n contains both α_n, β_n, our f_n arises from a strictly decreasing curve passing through the points $(\alpha_n, 1/n)$ and $(\beta_n, 1/2n)$. Let $f = \sum_n f_n$. The sum converges uniformly on X to a strictly positive function in $C_0(X)$. For each $\lambda > 1/n > 0$, we see that the level set $f^{-1}(\lambda)$ consists of at most $2n$ points in X. This is the required function we need in the first half of the proof. □

To end this paper, we would like to raise another problem.

Problem 3.5. Is every surjective linear local automorphism of a C*-algebra, or more generally, a semisimple Banach algebra, a Jordan isomorphism?

Remark that Crist [14] has an example of a bijective linear local automorphism of a three dimensional abelian radial subalgebra of the algebra M_3 of 3×3 matrices, which is not a Jordan homomorphism.

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