

# EXTENSION THEOREMS WITHOUT DEDEKIND COMPLETENESS

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ABSTRACT. In the operator version of the Hahn-Banach-Kantorovich theorem, the range space  $Y$  is assumed to be Dedekind complete. Y. A. Abramovich and A. W. Wickstead improved this by assuming only the Cantor property on  $Y$ . Along the same line of reasoning, we obtained in this paper several new results of this type. We also see that assuming Cantor property on the domain spaces instead gives good results, too.

## 1. INTRODUCTION

In this paper, all spaces are over the reals.

The Dedekind completeness assumption made on the range space  $Y$  in the operator version of the Hahn-Banach-Kantorovich theorem can be weakened when the domain space  $X$  is separable. Recall that an ordered linear space  $Y$  is said to have the *Cantor property* (or the  $(\sigma)$ -*interpolation property* or the *countable property*) if for every increasing sequence  $\{x_n\}_n$  and every decreasing sequence  $\{z_n\}_n$  in  $Y$  with  $x_n \leq z_n, \forall n = 1, 2, \dots$ , there is a  $y$  in  $Y$  such that  $x_n \leq y \leq z_n, \forall n = 1, 2, \dots$ . Y. A. Abramovich and A. W. Wickstead proved in [AW] the following Hahn-Banach type theorem.

**Theorem 1** ([AW]). *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $P : X \rightarrow Y_+$  be a continuous seminorm. Suppose  $G$  is a vector subspace of  $X$  and  $T : G \rightarrow Y$  is a continuous linear operator satisfying  $T(v) \leq P(v)$  for all  $v$  in  $G$ . Then there exists a continuous linear extension  $S$  of  $T$  to  $X$  such that  $S(x) \leq P(x)$  for all  $x$  in  $X$ .*

**Remark 2.** The argument in the proof of Theorem 1 in [AW] indeed works also for the case when  $X$  and  $Y$  are vector lattices and  $P : X \rightarrow Y_+$  is a continuous sublinear operator, i.e.  $P$  is subadditive and positively homogeneous.

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For a vector lattice we have: Dedekind completeness implies Dedekind  $(\sigma)$ -completeness implies Cantor property implies order completeness implies uniform completeness (see [Z, p. 696]). A first example of an ordered vector space  $X$  having the Cantor property is due to G. Seever (see [S]): Let  $K$  be a completely regular space. Then  $C(K)$  has the Cantor property if and only if  $K$  is an  $F$ -space (i.e. every pair of disjoint open  $(F_\sigma)$ -sets in  $K$  have disjoint closures). On the other hand, C. B. Huijsmans and B. de Pagter [HP] showed that an Archimedean vector lattice  $Y$  has the Cantor property if and only if  $Y$  is uniformly complete and normal. Recently, A. W. Wickstead [W] and N. Dăneț [DN] gave more equivalent conditions to the Cantor property (see Remark 9).

Recall that a vector subspace  $V$  of an ordered vector space  $X$  is a *majorizing subspace* if for every  $x$  in  $X$  there exists a  $v$  in  $V$  with  $x \leq v$ . Obviously, there also exists a  $u$  in  $V$  such that  $u \leq x$ . An  $e$  in  $X_+$  is said to be an *axial element* if the subspace of  $X$  spanned by  $e$  is majorizing. Moreover, positive linear operators from a majorizing subspace of a Banach lattice into a normed lattice are continuous. In [DW], the following Kantorovich type theorem was established.

**Theorem 3** ([DW]). *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property.*

1. *If  $V$  is a majorizing subspace of  $X$  and  $T : V \rightarrow Y$  is a positive linear operator, then  $T$  has a positive linear extension  $S : X \rightarrow Y$ .*
2. *If  $e$  is an axial element of  $X$ , then for each  $y_0$  in  $Y_+$  there exists a positive linear operator  $U : X \rightarrow Y$  with  $U(e) = y_0$ .*
3. *If the positive cone  $X_+$  of  $X$  has nonempty interior, then for each vector subspace  $G$  of  $X$  disjoint from  $\text{Int } X_+$  there exists a non-zero positive linear operator  $U : X \rightarrow Y$  vanishing on  $G$ .*

In this paper, we present some new extension theorems of this sort. In parallel to the classical Hahn-Banach-Kantorovich theorems, these results also

have interesting applications. In particular, we assume only the Cantor property on the range space, which is an interpolation property admittedly weaker than Dedekind completeness. We will also show that similar results hold when the domain space, rather than the range space, is assumed to have the Cantor property.

## 2. MAIN RESULTS

We begin with a Hahn-Banach type result.

**Corollary 4.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $P : X \rightarrow Y_+$  be a continuous sublinear operator. Then for any  $x_0$  in  $X$ , there exists a continuous linear operator  $U : X \rightarrow Y$  such that  $U(x_0) = P(x_0)$  and  $-P(-x) \leq U(x) \leq P(x)$  for all  $x$  in  $X$ .*

*Proof.* Let  $G$  be the subspace of  $X$  spanned by  $x_0$  and let  $T : G \rightarrow Y$  be defined by  $T(\lambda x_0) = \lambda P(x_0)$  for all  $\lambda$  in  $\mathbb{R}$ . Then  $T$  is a continuous linear operator such that  $T(v) \leq P(v)$  for all  $v$  in  $G$ . Now Theorem 1 (and Remark 2) applies.  $\square$

Recall that for vector lattices  $X$  and  $Y$ , a seminorm  $P : X \rightarrow Y_+$  is said to be a *lattice seminorm* if  $|x_1| \leq |x_2|$  in  $X$  implies  $P(x_1) \leq P(x_2)$ . Also, a seminorm (or a sublinear operator)  $P : X \rightarrow Y_+$  is called *monotone on  $X_+$*  if from  $0 \leq x_1 \leq x_2$  in  $X$  it follows  $P(x_1) \leq P(x_2)$ . Obviously, if  $P : X \rightarrow Y_+$  is a lattice seminorm, then  $P$  is monotone on  $X_+$  and  $P(x) = P(|x|)$  for all  $x$  in  $X$ . The converse is not true, in general. For example, consider the seminorm  $f \mapsto |\int f dx|$  of  $C[0, 1]$ .

In the proof of the following result, we use again Theorem 1 to extend positive linear operators.

**Theorem 5.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $P : X \rightarrow Y_+$  be a continuous sublinear operator. Suppose  $G$  is a vector sublattice of  $X$  and  $T : G \rightarrow Y$  is a positive linear operator such that  $T(v) \leq P(v), \forall v \in G$ . If  $P$  is monotone on  $X_+$ , then*

there exists a positive linear operator  $S : X \rightarrow Y$  extending  $T$  such that  $S(x) \leq P(x_+), \forall x \in X$ . In case  $P$  is a lattice seminorm, we can also assume  $S(x) \leq P(x), \forall x \in X$ .

*Proof.* Note that  $T$  is continuous due to being dominated by the continuous sublinear operator  $P$  on  $G$ . Define  $P_1 : X \rightarrow Y_+$  by  $P_1(x) = P(x_+)$  for all  $x$  in  $X$ . The monotonicity of  $P$  ensures that  $P_1$  is also a continuous sublinear operator. Since  $Tx \leq Tx_+ \leq P(x_+) = P_1(x)$  for all  $x$  in  $G$ , by Theorem 1 (and Remark 2) we obtain a continuous linear extension  $S : X \rightarrow Y$  of  $T$  such that  $S(x) \leq P_1(x) = P(x_+)$  for all  $x$  in  $X$ . We claim that  $S$  is positive. Indeed, if  $x \leq 0$  in  $X$  then  $P_1(x) = 0$  and hence  $S(x) \leq 0$ . Finally, we note that  $P(x_+) \leq P(|x|) = P(x)$  in case  $P$  is a lattice seminorm.  $\square$

The following two results are both very interesting consequences of Theorem 5. In the first one, employing an idea of P. Meyer-Nieberg [MN, p. 47] we will extend simultaneously a decreasing sequence of positive linear operators. It is clear that we can also obtain another version if we only assume  $P$  is a continuous sublinear operator monotone on  $X_+$ .

**Corollary 6.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $P_1 : X \rightarrow Y_+$  be a continuous lattice seminorm and let  $G$  be a vector sublattice of  $X$ . Suppose a sequence of positive linear operators  $T_n : G \rightarrow Y$  satisfies that  $T_1(v) \leq P_1(v)$  for all  $v$  in  $G$ , and  $T_{n+1}(v) \leq T_n(v)$  for all  $v$  in  $G_+$  and  $n = 1, 2, \dots$ . Then  $\{T_n\}_{n \geq 1}$  can be simultaneously extended to a sequence  $\{S_n\}_{n \geq 1}$  of positive linear operators  $S_n : X \rightarrow Y$  such that  $S_1(x) \leq P_1(x)$  for all  $x$  in  $X$ , and  $S_{n+1}(x) \leq S_n(x)$  for all  $x$  in  $X_+$  and  $n = 1, 2, \dots$ .*

*Proof.* By Theorem 5, we extend  $T_1$  to a positive linear operator  $S_1 : X \rightarrow Y$  such that  $S_1(x) \leq P_1(x)$  for all  $x$  in  $X$ . Define a continuous lattice seminorm  $P_2 : X \rightarrow Y_+$  by  $P_2(x) = S_1(|x|), \forall x \in X$ . Then  $T_2(v) \leq T_2(|v|) \leq T_1(|v|) = S_1(|v|) = P_2(v)$  for all  $v$  in  $G$ . By Theorem 5 again, we get a positive linear extension  $S_2 : X \rightarrow Y$  of  $T_2$  such that  $S_2(x) \leq P_2(x)$  for all  $x$  in  $X$ . For all  $x$

in  $X_+$ , we have  $S_2(x) \leq P_2(x) = S_1(|x|) = S_1(x)$ . We complete the proof by induction.  $\square$

The following result is another consequence of Theorem 5 and it also uses an idea of [MN, p. 46]. However, we assume here all the operators are *positive* and their common range spaces share only the Cantor property instead of the Dedekind completeness.

**Corollary 7.** *Let  $X$  and  $Y$  be two Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Suppose  $P_1, P_2, \dots, P_n : X \rightarrow Y_+$  are  $n$  continuous sublinear operators monotone on  $X_+$ , and  $T : X \rightarrow Y$  is a positive linear operator such that  $T(x) \leq P_1(x) + P_2(x) + \dots + P_n(x)$  for all  $x$  in  $X$ . Then there exist positive linear operators  $T_1, T_2, \dots, T_n : X \rightarrow Y$  such that  $T = T_1 + T_2 + \dots + T_n$ , and  $T_i(x) \leq P_i(x)$  for all  $i = 1, 2, \dots, n$  and  $x$  in  $X$ .*

*Proof.* Let  $Z$  be the canonical product space  $X^n$ . We remark that  $Z$  is a separable Banach lattice in the  $\ell_1$  norm. For any  $x$  in  $X$ , we denote by  $(\delta_{ij}(x))_{j=1}^n$ ,  $i = 1, 2, \dots, n$ , and  $z(x)$  the elements of  $Z$ , defined by

$$\delta_{ij}(x) = \begin{cases} x, & j = i, \\ 0, & j \neq i \end{cases}$$

for each  $i = 1, 2, \dots, n$ , and  $z(x) = \sum_{i=1}^n (\delta_{ij}(x))_{j=1}^n$ . In other words,  $z(x) = (x, x, \dots, x) \in Z$ . Obviously,  $G = \{z(x) \mid x \in X\}$  is a Banach sublattice of  $Z$ .

We define  $P : Z \rightarrow Y_+$  by  $P(x_1, x_2, \dots, x_n) = \sum_{i=1}^n P_i(x_i)$  for  $x_1, x_2, \dots, x_n$  in  $X$ , and  $T_0 : G \rightarrow Y$  by  $T_0(z(x)) = T(x)$  for  $x$  in  $X$ . It is easy to see that  $P$  is a continuous sublinear operator monotone on  $Z_+$ ,  $T_0$  is a positive linear operator and  $T_0(z(x)) \leq P(z(x))$  for all  $x$  in  $X$ . By Theorem 5, we can extend  $T_0$  to the whole  $Z$ , obtaining a positive linear operator  $S : Z \rightarrow Y$  such that  $S(x_1, x_2, \dots, x_n) \leq P(x_1, x_2, \dots, x_n)$  for all  $x_1, x_2, \dots, x_n$  in  $X$ .

For each  $i = 1, 2, \dots, n$  and  $x$  in  $X$ , we define  $T_i(x) = S((\delta_{ij}(x))_{j=1}^n)$ , and thus obtain  $n$  positive linear operators from  $X$  into  $Y$ . It then follows

$$T(x) = S(z(x)) = S\left(\sum_{i=1}^n (\delta_{ij}(x))_{j=1}^n\right) = \sum_{i=1}^n S((\delta_{ij}(x))_{j=1}^n) = \sum_{i=1}^n T_i(x)$$

and

$$T_i(x) = S((\delta_{ij}(x))_{j=1}^n) \leq P((\delta_{ij}(x))_{j=1}^n) = P_i(x).$$

□

**Corollary 8.** *Let  $X$  and  $Y$  be two Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $T, S_1, \dots, S_n : X \rightarrow Y$  be positive linear operators such that  $T(x) \leq S_1(x) + S_2(x) + \dots + S_n(x)$  for all  $x$  in  $X$ . Then there exist positive linear operators  $T_1, T_2, \dots, T_n : X \rightarrow Y$  such that  $T = T_1 + T_2 + \dots + T_n$  and  $T_i(x) \leq S_i(x)$  for all  $i = 1, 2, \dots, n$  and  $x$  in  $X$ .*

*Proof.* Set  $P_i(x) = S_i(|x|)$  for all  $x$  in  $X$  and  $i = 1, 2, \dots, n$ . Then Corollary 7 applies. □

**Remark 9.** Recall that a linear operator  $T$  between ordered vector spaces is called *regular* if  $T = T_1 - T_2$  is the difference of two positive linear operators. We denote by  $R(X, Y)$  the space of all regular operators from  $X$  into  $Y$ . In case  $Y$  is a Banach lattice, A. W. Wickstead [W] proved that the following are all equivalent. Here,  $c$  denotes the Banach lattice of convergent sequences.

- (i)  $Y$  has the Cantor property.
- (ii) The space  $R(c, Y)$  has the strong  $(\sigma)$ -interpolation property.
- (iii) The space  $R(c, Y)$  has the Riesz decomposition property.

More recently, N. Dăneţ [DN] showed that they are also equivalent to:

- (iii') The space  $R(X, Y)$  has the Riesz decomposition property for any separable Banach lattice  $X$ .

In [DN], the main part of the proof is “(i) $\Rightarrow$ (iii)’”. We remark that this can also be obtained by a direct application of Corollary 8.

Recall that a subset  $A$  of  $Y$  is said to be *order bounded* or *majorized* (from above by a  $u$  in  $Y_+$ ) if  $a \leq u$  for all  $a$  in  $A$ . The following result is inspired by [C2, p. 332].

**Theorem 10.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable and  $Y$  has the Cantor property. Let  $B$  be the closed unit ball of a vector subspace*

$G$  of  $X$ . Let  $T : G \rightarrow Y$  be a positive linear operator. Then  $T$  has a positive linear extension to  $X$  if  $T(B)$  is order bounded. When  $Y$  has an axial element,  $T$  has a positive linear extension to  $X$  if and only if  $T(B)$  is norm bounded.

*Proof.* Let  $M = B - X_+$  be the translate of  $B$  by  $-X_+$ , i.e.

$$M = \{x \in X \mid x \leq b \text{ for some } b \text{ in } B\}.$$

Let  $T(B)$  be majorized by  $y_0$  in  $Y_+$ . Then  $T(G \cap M)$  is majorized by  $y_0$ , too. Let  $p_M$  be the continuous sublinear Minkowski functional of the convex zero-neighborhood  $M$ . Let  $P : X \rightarrow Y_+$  be the sublinear operator defined by  $P(x) = p_M(x)y_0$ . For every  $v$  in  $G$ , since  $v \in (p_M(v) + \varepsilon)(G \cap M)$  for all  $\varepsilon > 0$ , we have  $T(v) \leq P(v)$ .

It follows from Theorem 1 (and Remark 2), we can find a linear operator  $S : X \rightarrow Y$  extending  $T$  such that  $S(x) \leq P(x)$  for all  $x$  in  $X$ . We claim that  $S$  is also positive. In fact, if  $x$  is in  $X_+$  and  $n$  is a positive integer, then  $-nx = 0 - nx \in M$ , and hence  $p_M(-nx) \leq 1$ . It then gives  $S(-nx) \leq y_0$ , and thus  $S(x) \geq -\frac{1}{n}y_0$  for all  $n = 1, 2, \dots$ . Hence,  $S(x) \geq 0$ . Consequently,  $S$  is positive and thus continuous.

Finally, we suppose there exists an axial element  $w$  in  $Y$ . If  $S : X \rightarrow Y$  is a positive linear extension of  $T$ , then the set

$$A = \{x \in X : |S(x)| \leq w\}$$

is a closed, absorbing convex subset of  $X$ . By the Baire Category Theorem, there is a positive scalar  $\lambda$  such that  $B \subseteq \lambda A$ . Thus,  $T(B)$  is majorized by  $y_0 = \lambda w$  and thus norm bounded. Conversely, we can assume  $G$  is closed if  $T$  is bounded. Then

$$A' = \{x \in G : |T(x)| \leq w\}$$

is a closed, absorbing convex subset of  $G$ . By the Baire Category Theorem again,  $A'$  contains a multiple of the closed unit ball  $B$  of  $G$ . In particular,  $T(B)$  is majorized and the first part of the proof applies.  $\square$

## 3. ASSUMING CANTOR PROPERTY ON DOMAIN SPACES

In this section, we try to relax the assumption on the range space  $Y$ .

**Corollary 11.** *Let  $X$  and  $Y$  be two Banach lattices such that  $X$  is separable. Let  $G$  be a majorizing vector sublattice of  $X$  and let  $G$  have the Cantor property. Then every positive linear operator  $T : G \rightarrow Y$  can be extended to a positive linear operator  $S : X \rightarrow Y$ .*

*Proof.* By the proof of Theorem 3 in [DW], we see that the identity mapping  $1_G : G \rightarrow G$  can be extended to a positive linear operator  $U : X \rightarrow G$ . Then  $S = T \circ U$  is a positive linear operator from  $X$  into  $Y$  and obviously  $S = T$  on  $G$ . □

**Remark 12.** The above proof shows that for every majorizing vector sublattice  $G$  of a separable Banach lattice  $X$ , if  $G$  has the Cantor property then  $G$  is positively complemented in  $X$ . In other words, there exists a continuous positive projection  $S$  from  $X$  onto  $G$ .

**Corollary 13.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  is separable. Let  $G$  be a vector sublattice of  $X$  and let  $G$  have the Cantor property. Suppose  $P : X \rightarrow G_+$  is a continuous sublinear operator monotone on  $X_+$  such that  $v \leq P(v)$  for all  $v$  in  $G$ . If  $T : G \rightarrow Y$  is a continuous (resp. positive) linear operator, then  $T$  can be extended to a continuous (resp. positive) linear operator  $S : X \rightarrow Y$ .*

*Proof.* The condition  $v \leq P(v)$  for all  $v$  in  $G$  can be written as  $1_G(v) \leq P(v)$  for all  $v$  in  $G$ . By Theorem 5, the identity map  $1_G : G \rightarrow G$  can be extended to a continuous positive linear operator  $U : X \rightarrow G$ . If  $T$  is continuous or positive, then so is the extension  $S = T \circ U$ . □

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