LINEAR ORTHOGONALITY PRESERVERS OF HILBERT C^*-MODULES OVER C^*-ALGEBRAS WITH REAL RANK ZERO

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Abstract. Let A be a C^*-algebra. Let E and F be Hilbert A-modules with E being full. Suppose that $\theta : E \to F$ is a linear map preserving orthogonality, i.e.,

$$\langle\theta(x), \theta(y)\rangle = 0 \quad \text{whenever} \quad \langle x, y\rangle = 0.$$

We show in this article that if, in addition, A has real rank zero, and θ is an A-module map (not assumed to be bounded), then there exists a central positive multiplier $u \in M(A)$ such that

$$\langle\theta(x), \theta(y)\rangle = u\langle x, y\rangle \qquad (x, y \in E).$$

In the case when A is a standard C^*-algebra, when A is a properly infinite unital C^*-algebra, or when A is a W^*-algebra, we also get the same conclusion with the assumption of θ being an A-module map weakened to being a local map.

1. Introduction and Notations

It is a common knowledge that, together with the linearity, the inner product and the norm structures of a Hilbert space H determine each other. It might be a bit less well-known that the orthogonality structure also suffices to determine the inner product up to a scalar. This fact follows from the following easy observation: for any $x, y \in H$, $\|x\| = \|y\|$ if and only if $x + \lambda y$ is orthogonal to $x - \lambda y$ for some scalar λ with $|\lambda| = 1$ (see also [3, 6]).

It is natural and interesting to ask whether the linear structure and orthogonality structure of a (complex) Hilbert C^*-module determines its C^*-algebra-valued inner product. More precisely, let A be a (complex) C^*-algebra, and $\theta : E \to F$ be a $\mathbb{C}$-linear map between Hilbert A-modules that preserves orthogonality (i.e. preserves zero A-valued inner products). We want to study to what extent, θ will respect the A-valued inner products. When the underlying C^*-algebra is $\mathbb{C}$, it reduces to the case of Hilbert spaces.

We first note that without any further assumption on θ, the above question might have a negative answer.

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Example 1.1. Let H be an infinite dimensional (complex) Hilbert space and $A = \mathcal{K}(H)$ be the C^*-algebra of all compact operators on H. Suppose that $\bar{H}$ is a vector space that is conjugate-linear isomorphic to H. When equipped with the operations: $\langle \overline{\eta_1}, \overline{\eta_2} \rangle := \eta_1 \otimes \eta_2$ and $\overline{\eta_1} T := \overline{T^* \eta_1}$ ($\overline{\eta_1}, \overline{\eta_2} \in \bar{H}; T \in A$), we see that $\bar{H}$ is a Hilbert A-module. Suppose that θ is any unbounded bijective $\mathbb{C}$-linear map from $\bar{H}$ onto $\bar{H}$. Since $\langle x, y \rangle = 0$ if and only if $x = 0$ or $y = 0$, we see that both θ and θ^{-1} preserves orthogonality.

As we are dealing with Hilbert A-modules, a natural additional assumption is that θ is an A-module map, i.e., $\theta(xa) = \theta(x)a$ ($x \in E, a \in A$). In [9], Ilišević and Turnšek showed that if A is a standard C^*-algebra, then for every orthogonality preserving A-module map $\theta : E \to F$, there is a scalar $\lambda \geq 0$ such that $\langle \theta(x), \theta(y) \rangle = \lambda \langle x, y \rangle$ ($x, y \in E$). In particular, all such θ are scalar multiples of isometries.

In [13], under a weaker assumption on θ, namely θ being "local", we get the same conclusion in the case when A is a commutative C^*-algebra (in fact, the main difficulties in [13] come from the fact that θ is not assumed to be an A-module map). Recall that a $\mathbb{C}$-linear map $\theta : E \to F$ is *local* if

$$\theta(x)a = 0 \quad \text{whenever} \quad xa = 0 \qquad (x \in E; a \in A).$$

Readers should find the idea of local linear maps familiar. For example, local linear maps on the space of smooth functions defined on a manifold modeled on $\mathbb{R}^n$ are exactly linear differential operators (see, e.g., [19, 16]). See [11, 2] for the vector-valued case, and [1] for the Banach $C^1[0,1]$-module setting. We also mention that there is a bimodule version of local maps as studied by Schwizer in [20] (which is different from ours). Notice that every module map is local, but local linear maps, e.g., linear differential operators, might not be a module map. Nevertheless, it has been shown in [12, Proposition A.1] that every *bounded* local map between Hilbert C^*-modules is a module map.

The results in [9] and [13] lead to the following conjecture. We remark that the fullness assumption of E in this conjecture is a necessity. Without this, the conclusion does not hold even in the case when A is commutative (see [13, 3.6]). Here, a Hilbert A-module E is said to be *full* if the linear span of $\{\langle x, y \rangle : x, y \in E\}$ is dense in A.

Conjecture 1.2. *Let A be a C^*-algebra. Let E and F be Hilbert A-modules with E being full. If $\theta : E \to F$ is a ($\mathbb{C}$-linear) local map preserving orthogonality, i.e. for any $x, y \in E$,*

$$\langle x, y \rangle = 0 \quad \textit{implies} \quad \langle \theta(x), \theta(y) \rangle = 0,$$

then there is a central positive element $u \in M(A)$ such that

$$\langle \theta(x), \theta(y) \rangle = u \langle x, y \rangle \qquad (x, y \in E).$$

In this article, positive answers of this conjecture will be given in the following four cases:

(1) A is a C^*-algebra of real rank zero and θ is an A-module map (Theorem 2.3).

(2) A is a standard C^*-algebra (Corollary 3.2).
(3) A is a properly infinite unital C^*-algebra (Corollary 3.3).
(4) A is a W^*-algebra (Corollary 3.4).

As a final remark for the introduction, we note that, unlike the situation in some other literatures (e.g. [7]), θ is *not* assumed to be bounded, for the conceptual reasons as stated in the beginning of this section (but whose boundedness will be an automatic consequence of our results).

Let us now give some notations that will be used throughout this article. In the following, A is a C^*-algebra, E and F are Hilbert A-modules, and $\Psi, \theta : E \to F$ are orthogonality preserving $\mathbb{C}$-linear maps, which are not assumed to be bounded.

Let $a \in A_+$. We set $C^*(a)$ to be the C^*-subalgebra generated by a, and $\mathbf{c}(a)$ to be the *central cover* of a, i.e., the smallest central element in A^{**}_+ dominating a (see, e.g., [18, 2.6.2]). If, in addition, $\alpha, \beta \in \mathbb{R}_+$, we put $e_a(\alpha, \beta]$ to be the spectral projections of a in A^{**} corresponding to the set $(\alpha, \beta] \cap \sigma(a)$.

We denote by $Z(A)$ the center and by $M(A)$ the space of all multipliers of A. On the other hand, $\mathrm{Proj}_1(A)$ is the set of all norm-one (i.e., non-zero) projections in A. For any *open projection* $p \in \mathrm{Proj}_1(A^{**})$ (i.e., there exists an increasing net $\{a_i\}$ in A_+ such that $a_i \uparrow p$ in the weak-*-topology), we denote by $\mathrm{her}(p) := pA^{**}p \cap A$ the hereditary C^*-subalgebra associated to p. See, e.g., [4] for more information about open projections.

2. Orthogonality preserving A-module maps when A has real rank zero

We recall that A has *real rank zero* if every self-adjoint element in A can be approximated in norm by invertible self-adjoint elements. This is equivalent to the real linear span of $\mathrm{Proj}_1(A)$ being norm dense in A_{sa} (see, e.g., [5]).

Let us start with the following easy lemma. Part (a) of which might be well-known but we give an argument here for completeness.

Lemma 2.1. *(a) If $p \in \mathrm{Proj}_1(A^{**})$ and $b \in Z(pA^{**}p)_+$, then $\|\mathbf{c}(b)\| = \|b\|$, $\mathbf{c}(b)p = b$ and $\mathbf{c}(b)\mathbf{c}(p) = \mathbf{c}(b)$.*

(b) Suppose that A has real rank zero and E is full. If $q \in A^{**} \setminus \{0\}$ is an open projection, there are $r \in \mathrm{Proj}_1(A)$ and $y \in Er$ such that $r = \langle y, y\rangle \leq q$.

Proof. (a) Since $b \leq \|b\|1$, we see that $0 \leq b \leq \mathbf{c}(b) \leq \|b\|1$ and $\|\mathbf{c}(b)\| = \|b\|$. Clearly, $\mathbf{c}(b)p = p\mathbf{c}(b)p \geq pbp = b$. Conversely, as $Z(pA^{**}p) = Z(A^{**})p$ (see e.g. [10, 5.5.6]), there is $a \in Z(A^{**})_+$ with $b = ap$ (note that $b^{1/2} \in Z(A^{**})p$). Thus, we have $b = a^{1/2}pa^{1/2} \leq a^{1/2}\mathbf{c}(p)a^{1/2} = a\mathbf{c}(p)$. As $a\mathbf{c}(p)$ is central, $\mathbf{c}(b) \leq a\mathbf{c}(p)$ and $\mathbf{c}(b)p = p\mathbf{c}(b)p \leq ap = b$. The last equality follows from [18, 2.6.4] and the fact that $b\mathbf{c}(p) = b$.

(b) Note that $\mathrm{her}(q) \neq (0)$, and also has real rank zero (see e.g. [5]). Moreover, $E_0 := E\cdot\mathrm{her}(q)$ is a full (and, hence non-zero) Hilbert $\mathrm{her}(q)$-module. Pick any $x \in E_0$ such that $a := \langle x, x\rangle$ is a norm one element in $\mathrm{her}(q)$. Let $t \in (0, 1/3)$ and $b \in \mathrm{her}(q)_+$ such that $\|a-b\| < t$ and $\sigma(b) = \{\lambda_1, ..., \lambda_n\}$ with $\lambda_1 \leq \cdots \leq \lambda_n = \|b\|$ (see e.g. [5]). Since $\|b\| > 2/3$, we can choose $s \in [t, \|b\|] \setminus \sigma(b)$. If we set $r := e_b(s, 2] \in \mathrm{Proj}_1(A)$, then $\|r - rar\| \leq \|r - rbr\| + \|b - a\| < 1$. If $c := r + \sum_{n=1}^{\infty}(r - rar)^n \in A_+$, then $(rar)c = c(rar) = r$ and so, $\langle xc^{1/2}, xc^{1/2}\rangle = c^{1/2}rarc^{1/2} = r$. Finally, $xc^{1/2} \in Er$ as $c^{1/2}$ is in the C^*-subalgebra $rAr + \mathbb{C}r$. □

Proposition 2.2. *Let A be a unital C^*-algebra of real rank zero. Suppose that $\theta : E \to F$ is an A-module map preserving orthogonality, and there is an element $x_0 \in E$ such that $\langle x_0, x_0\rangle = 1$. Then one can find $u \in Z(A)_+$ satisfying*

$$\langle\theta(x), \theta(y)\rangle \;=\; u\langle x, y\rangle \quad (x, y \in E).$$

Proof. Let $u := \langle\theta(x_0), \theta(x_0)\rangle \in A_+$. For any symmetry $w \in A$, as $x_0 + x_0w$ and $x_0 - x_0w$ are orthogonal to each other, so are $\theta(x_0) + \theta(x_0)w$ and $\theta(x_0) - \theta(x_0)w$. Consequently, $u + wu - uw - wuw = 0$ and $u + uw - wu - wuw = 0$ (by taking adjoint). This tells us that $u = wuw$, and so, $u \in Z(A)_+$ (as A is generated by projections, and thus also by symmetries). Pick any $z \in E$ with $\langle x_0, z\rangle = 0$. Then $z + x_0\langle z, z\rangle^{1/2}$ is also orthogonal to $z - x_0\langle z, z\rangle^{1/2}$. It follows from the orthogonality preserving property that

$$\langle\theta(z), \theta(z)\rangle \;=\; \langle z, z\rangle^{1/2}\langle\theta(x_0), \theta(x_0)\rangle\langle z, z\rangle^{1/2} \;=\; u\langle z, z\rangle.$$

For any $y \in E$, the element $z = y - x_0\langle x_0, y\rangle$ is orthogonal to x_0. Hence,

$$\langle\theta(y), \theta(y)\rangle \;=\; \langle y, x_0\rangle\langle\theta(x_0), \theta(x_0)\rangle\langle x_0, y\rangle + \langle\theta(z), \theta(z)\rangle \;=\; u\langle y, y\rangle.$$

A polarization type argument implies that $\langle\theta(x), \theta(y)\rangle = u\langle x, y\rangle$ $(x, y \in E)$. □

Theorem 2.3. *Let A be a C^*-algebra of real rank zero. Suppose that E is full, and $\theta : E \to F$ is an orthogonality preserving A-module map (not assumed to be bounded). There is $u \in Z(M(A))_+$ such that*

$$\langle\theta(x), \theta(y)\rangle \;=\; u\langle x, y\rangle \quad (x, y \in E).$$

In particular, θ is automatically bounded.

Proof. Set

$$P \;:=\; \{(x, p) \in E \times \mathrm{Proj}_1(A) : \langle x, x\rangle = p \text{ and } xp = x\}.$$

Lemma 2.1(b) tells us that $P \neq \emptyset$. Suppose that $(x, p) \in P$. Then Ep is a full Hilbert pAp-module and the restriction of θ on Ep is an orthogonality preserving pAp-module map. Since p is the identity of the C^*-algebra pAp and $\theta(Ep) \subseteq Fp$, one can apply Proposition 2.2 to obtain $b_p \in Z(pAp)_+$ that satisfies

$$\langle\theta(x)p, \theta(y)p\rangle \;=\; b_p\langle xp, yp\rangle \qquad (x, y \in E).$$

By Lemma 2.1(a), we have

$$p\left(\langle\theta(x), \theta(y)\rangle - \mathbf{c}(b_p)\langle x, y\rangle\right)p \;=\; 0 \qquad (x, y \in E). \tag{2.1}$$

As the weak-*-closed linear span, I, of $\{\langle\theta(x),\theta(y)\rangle - \mathbf{c}(b_p)\langle x,y\rangle : x,y \in E\}$ is an ideal of A^{**}, there is a central projection $q_I \in A^{**}$ with $I = q_I A^{**}$. Since $pq_I = pq_I p = 0$, we have $\mathbf{c}(p) \leq 1 - q_I$. Consequently,

$$\mathbf{c}(p)\langle\theta(x),\theta(y)\rangle \ = \ \mathbf{c}(p)\mathbf{c}(b_p)\langle x,y\rangle \qquad (x,y \in E). \tag{2.2}$$

Now, let $\mathcal{D} := \{D \subseteq P : \mathbf{c}(p)\mathbf{c}(q) = 0 \text{ whenever } (x,p),(y,q) \in D\}$. If we equip $\mathcal{D}$ with the usual inclusion, then Zorn's Lemma gives a maximal element $D_0 = \{(x_\gamma, p_\gamma)\}_{\gamma\in\Gamma} \in \mathcal{D}$. Set $q_0 := \bigvee_{\gamma\in\Gamma} \mathbf{c}(p_\gamma)$ in $\mathrm{Proj}_1(A^{**})$, which is a central element. Observes that $1 - q_0$ will not dominate a non-trivial open projection. Indeed, if $0 \neq q \leq 1 - q_0$ is an open projection, then Lemma 2.1(b) produces an element $(y,r) \in P$ such that $r \leq q$. Therefore, $D_0 \cup \{(y,r)\} \in \mathcal{D}$ which contradicts the maximality of D_0. We now claim that the $*$-homomorphism $\Phi : A \to q_0 A \subseteq A^{**}$ defined by $\Phi(a) = q_0 a$ is injective. Suppose on the contrary that there exists $a \in A_+$ with $\|a\| = 1$ and $\Phi(a) = 0$. Take any $\epsilon \in (0,1)$, and put q_ϵ to be the non-zero open projection $e_a(\epsilon, 1]$. As $a - \epsilon q_\epsilon \geq 0$, we have $q_\epsilon q_0 = q_0 q_\epsilon q_0 \leq q_0 a q_0/\epsilon = 0$. So, $q_\epsilon \leq 1 - q_0$ which implies the contradiction that $q_\epsilon = 0$.

As $x_\gamma \mathbf{c}(p_\gamma) = x_\gamma p_\gamma \mathbf{c}(p_\gamma) = x_\gamma$ $(\gamma \in \Gamma)$, we see that x_γ and $x_{\gamma'}$ are orthogonal if $\gamma \neq \gamma'$. We now claim that $\mathbf{c}(b_\gamma)$'s are uniformly bounded (where $b_\gamma \in Z(p_\gamma A p_\gamma)_+$ is the element associated with $(x_\gamma, p_\gamma) \in P$ that satisfies (2.2)). Suppose on the contrary that there are $\mathbf{c}(b_{\gamma_n})$ with $\|\mathbf{c}(b_{\gamma_n})\| = \|b_{\gamma_n}\| \geq n^3$ $(n \in \mathbb{N})$. Note that the orthogonal sum $x := \sum_n \frac{x_{\gamma_n}}{n}$ convergent in norm in E. By the orthogonality preserving property of θ, Lemma 2.1(a) as well as Equality (2.1), for any $m \in \mathbb{N}$,

$$\begin{aligned}\langle\theta(x),\theta(x)\rangle &= \ \langle\theta\left(x_{\gamma_m}/m\right),\theta\left(x_{\gamma_m}/m\right)\rangle + \langle\theta\left(x - x_{\gamma_m}/m\right),\theta\left(x - x_{\gamma_m}/m\right)\rangle \\ &\geq \ \langle\theta\left(x_{\gamma_m}/m\right),\theta\left(x_{\gamma_m}/m\right)\rangle \ = \ \frac{\mathbf{c}(b_{\gamma_m})\left\langle x_{\gamma_m}, x_{\gamma_m}\right\rangle}{m^2} \ = \ \frac{b_{\gamma_m}}{m^2}.\end{aligned}$$

As the norm of the last term goes to infinity as $n \to \infty$, we reach a contradiction.

Finally, let d be the weak-*-limit in A^{**} of finite sums of the uniformly bounded mutually orthogonal elements $\mathbf{c}(b_\gamma)$ (see Lemma 2.1(a)). By Relation (2.2) and the fact that q_0 is the weak-*-limit of finite sums of $\mathbf{c}(p_\gamma)$'s, we have

$$dq_0\langle x,y\rangle \ = \ q_0\langle\theta(x),\theta(y)\rangle \ \in \ q_0 A \qquad (x,y \in E).$$

Since E is full, we see that d induces an element $m \in Z(M(q_0 A))_+$ such that $mq_0\langle x,y\rangle = q_0\langle\theta(x),\theta(y)\rangle$ $(x,y \in E)$. Since $\Phi : A \to q_0 A$ extends to a $*$-isomorphism $\tilde{\Phi} : M(A) \to M(q_0 A)$, there is $u \in Z(M(A))_+$ such that $\tilde{\Phi}(u) = m$. This means that

$$\Phi(u\langle x,y\rangle - \langle\theta(x),\theta(y)\rangle) = 0 \qquad (x,y \in E)$$

which gives the required conclusion. □

Remark 2.4. Let A be a general C^*-algebra. Suppose that there exist Hilbert A^{**}-modules $\tilde{E}$ and $\tilde{F}$ containing E and F respectively, such that the Hilbert A^{**}-module structures extend the corresponding Hilbert A-module structures, and that θ extends to an orthogonality preserving A^{**}-module map $\tilde{\theta} : \tilde{E} \to \tilde{F}$. Then one can use Theorem 2.3 to show that θ satisfies the conclusion of Conjecture 1.2 (since A^{**} has real rank zero). In the situation when θ is a bounded orthogonality preserving A-module map, we have tried $\tilde{E} = E^{**}$ and $\tilde{F} = F^{**}$ but encountered some difficulties in showing that θ^{**} is orthogonality preserving. It was claimed in [7] that, when θ is bounded, such $\tilde{E}$, $\tilde{F}$ and $\tilde{\theta}$ could be found. However, instead of manipulating the difficulties in the arguments in [7], we are working on a proof for the case of general C^*-algebras, *without* the boundedness assumption on θ (but θ is assumed to be a A-module map), using completely different ideas from those in this article, in [7], in [9], nor in [13].

3. Orthogonality preserving $\mathbb{C}$-linear local maps

In this section, we consider ($\mathbb{C}$-linear) local maps (see the Introduction) that preserve orthogonality. Let us first give the following useful observation.

Lemma 3.1. *Let A_0 be the $*$-algebra generated by all the idempotents in A. If $\Psi : E \to F$ is a local map, then Ψ is an A_0-module map.*

Proof. Let $p \in A$ be an idempotent and $x \in E$. As Ψ is local, one has $\Psi(x-xp)p = 0$. If $\{u_i\}$ is an approximate unit for A, then $(1-p)u_i \in A$ will strictly converge to $(1-p)$. Since $\Psi(xp)(1-p)u_i = 0$, we have $\langle y, \Psi(xp)\rangle(1-p) = \lim \langle y, \Psi(xp)\rangle(1-p)u_i = 0$ $(y \in F)$. This implies that $\Psi(xp) - \Psi(xp)p = \Psi(xp)(1-p) = 0$. Thus, $\Psi(x)p = \Psi(xp)$, and so $\Psi(xv) = \Psi(x)v$ for any $v \in A_0$. □

Note that if A has real rank zero, then A_0 is dense in A. We remark however that A_0 can be $\{0\}$ (e.g. if $A = C_0(0,1)$).

Recall that A is a *standard C^*-algebra* on a Hilbert space H if $\mathcal{K}(H) \subseteq A \subseteq \mathcal{L}(H)$. In this case, A_0 contains a "big enough" ideal $\mathcal{F}(H)$ of A, in the sense that $\mathcal{K}(H) = \overline{\mathcal{F}(H)}$ is an essential ideal. As a consequence, we can use Lemma 3.1 and Theorem 2.3 to give a self-contained proof of the following slight extension of [9, 2.3] (note that the A-linearity is replaced by the local property).

Corollary 3.2. *(c.f.* [9, 2.3]*) Suppose that A is a standard C^*-algebra on a Hilbert space H. If $\Psi : E \to F$ is local and orthogonality preserving, then there is $\lambda \in \mathbb{R}_+$ such that*

$$\langle \Psi(x), \Psi(y)\rangle \;=\; \lambda\langle x, y\rangle \qquad (x, y \in E).$$

Proof. Consider $E_0 := E \cdot \mathcal{K}(H)$ and $F_0 := F \cdot \mathcal{K}(H)$ (both of them being Hilbert $\mathcal{K}(H)$-modules). Let $\{v_\gamma\}_{\gamma\in\Gamma}$ be an approximate unit in $\mathcal{K}(H)$ consisting of finite rank positive operators. By Lemma 3.1, $\Psi(xv) = \Psi(x)v$ for every finite rank operator v and every $x \in E$. On the other hand, for any $y \in E_0$, there exist $a \in \mathcal{K}(H)$ and $x \in E_0$ with $y = xa$ (by the Cohen factorization theorem), and so,

$$\Psi(y)v_\gamma \;=\; \Psi(xav_\gamma) \;=\; \Psi(x)av_\gamma \quad (\gamma \in \Gamma)$$

which shows that $\|\Psi(y)v_\gamma - \Psi(x)a\| \to 0$ (along γ). Define $\Phi : E_0 \to F_0$ by setting $\Phi(y)$ to be the norm limit of $\Psi(y)v_\gamma$. As $\Phi(y) = \Psi(x)a$ as well, we see that $\Phi(y)$

does not depend on the choice of $\{v_\gamma\}_{\gamma\in\Gamma}$, nor on the decomposition $y = xa$. If $b \in \mathcal{K}(H)$, then

$$\Phi(yb) = \Phi(xab) = \Psi(x)ab = \Phi(y)b.$$

Moreover, if $x, y \in E_0$ with $\langle x, y\rangle = 0$, then $\langle \Psi(x), \Psi(y)\rangle = 0$ which implies that $\langle \Psi(x)v_\gamma, \Psi(y)v_{\gamma'}\rangle = 0$ $(\gamma, \gamma' \in \Gamma)$, and so, $\langle \Phi(x), \Phi(y)\rangle = 0$. On the other hand, since $\mathcal{K}(H)$ is simple, we see that either E_0 is a full $\mathcal{K}(H)$-module or $E_0 = \{0\}$. By Theorem 2.3, there exists $\lambda \in Z(M(\mathcal{K}(H)))_+ = \mathbb{R}_+$ such that for every $x, y \in E_0$, one has $\langle \Phi(x), \Phi(y)\rangle = \lambda\langle x, y\rangle$ (note that one can take any λ if $E_0 = \{0\}$). Thus, for any $x, y \in E$ and $\gamma, \gamma' \in \Gamma$,

$$v_\gamma\langle \Psi(x), \Psi(y)\rangle v_{\gamma'} = \langle \Phi(xv_\gamma), \Phi(yv_{\gamma'})\rangle = \lambda\langle xv_\gamma, yv_{\gamma'}\rangle = \lambda v_\gamma\langle x, y\rangle v_{\gamma'}.$$

Consequently, if $b_{x,y} := \langle \Psi(x), \Psi(y)\rangle - \lambda\langle x, y\rangle \in A$, then $v_\gamma b_{x,y} v_{\gamma'} = 0$ $(\gamma, \gamma' \in \Gamma)$, which show that $b_{x,y} = 0$ (as $v_\gamma \to 1$ in the strong operator topology). □

We recall that a unital C^*-algebra A is said to be *properly infinite* if there exists $p \in \mathrm{Proj}_1(A)$ such that $p \sim 1 \sim 1 - p$.

Corollary 3.3. *Let A be a properly infinite unital C^*-algebra. If E is full and $\Psi : E \to F$ is an orthogonality preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y)\rangle = u\langle x, y\rangle \qquad (x, y \in E).$$

Proof. Let A_1 be the linear span of projections in A. By [15, Corollary 2.2], we see that $A = A_1$. Therefore, Lemma 3.1 tells us that Ψ is an A-module map. On the other hand, as $A = A_1$, we see that A has real-rank zero, and we can apply Theorem 2.3 to obtain the conclusion. □

Corollary 3.4. *Let A be a W^*-algebra. If E is full and $\Psi : E \to F$ is an orthogonality preserving local map, then there is $u \in Z(A)_+$ such that*

$$\langle \Psi(x), \Psi(y)\rangle = u\langle x, y\rangle \qquad (x, y \in E).$$

Proof. By Theorem 2.3, it suffices to show that Ψ is an A-module map. Recall that there are mutually orthogonal central projections q_{11}, q_{21} and q_∞ in A summing up to 1 such that $q_{11}A$ is a finite W^*-algebra of type I, $q_{21}A$ is a finite W^*-algebra of type II, and $q_\infty A$ is a properly infinite W^*-algebra (see, e.g., [14, 6.1.9]). Thus, $E = Eq_{11} \oplus Eq_{21} \oplus Eq_\infty$. The restriction $\Psi|_{Eq_\infty}$ is an $(q_\infty A)$-module map because of Lemma 3.1 and the fact that every element in $q_\infty A$ is a sum of at most five idempotents (see [17, Theorem 4]). Similarly, the restriction $\Psi|_{Eq_{21}}$ is an $(q_{21}A)$-module map since every element in $q_{21}A$ is a complex linear combination of at most twenty-four projections [8, Theorem 2]. Thus, it remains to verify the case when A is a finite W^*-algebra of type I.

In this case, for each $n \in \mathbb{N}$, there exist a hyperstonean space Ω_n (could be empty) and a projection $q_n \in Z(A)$ such that $\{q_n\}$ are orthogonal to one another, $\sum_n q_n$ weak-*-converges 1 and $q_nA \cong C(\Omega_n) \otimes M_n$ (see e.g. [14, 6.7.7]). Here we use the convention that $C(\Omega_n) = \{0\}$ if $\Omega_n = \emptyset$. Let $n \in \mathbb{N}$ such that $\Omega_n \neq \emptyset$ and $e \in C(\Omega_n)$

be the identity. Pick any rank one projection $p \in M_n$. If $r := e \otimes p \in \mathrm{Proj}_1(q_nA)$, then rAr is isomorphic to $C(\Omega_n)$. By Lemma 3.1, the induced map $\Psi_r : Er \to Fr$ is an orthogonality preserving local map between Hilbert rAr-modules. Using [13, 3.5], we see that Ψ_r is a rAr-module map. In particular, for any $a \in C(\Omega_n)$ and $x \in E$, one has

$$\Psi(x(a \otimes p)) \ = \ \Psi_r(xr(a \otimes I)r) \ = \ \Psi_r(xr)r(a \otimes I)r \ = \ \Psi(x)(a \otimes p),$$

where $I \in M_n$ is the identity. Now, let $\{e_{kl}\}_{k,l=1}^n$ be the matrix unit of M_n. As $\frac{e_{kl}+e_{kl}^*+e_{kk}+e_{ll}}{2}$ and $\frac{\mathrm{i}(e_{kl}-e_{kl}^*)+e_{kk}+e_{ll}}{2}$ are rank one projections, we see that e_{kl} is a linear combinations of rank one projections ($k, l \in \{1, ..., n\}$). Since any $a \in q_nA$ is of the form $a = \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ ($a_{ij} \in C(\Omega_n)$), we see that $\Psi(xa) = \Psi(x)a$ ($x \in E; a \in q_nA$). It follows that for any $x \in E$ and $a \in A$,

$$\Psi(xa)\sum_{k=1}^n q_k \ = \ \Psi\left(xa\sum_{k=1}^n q_k\right) \ = \ \Psi(x)\left(a\sum_{k=1}^n q_k\right) \ = \ (\Psi(x)a)\sum_{k=1}^n q_k \qquad (n \in \mathbb{N}).$$

Consequently, for any $y \in F$, we have $\langle y, \Psi(xa) - \Psi(x)a\rangle \sum_{k=1}^n q_k = 0$ which implies that $\langle y, \Psi(xa) - \Psi(x)a\rangle = 0$, and so $\Psi(xa) = \Psi(x)a$ as required. □

We end this article with a result that could be a first step towards a positive answer for Conjecture 1.2 in the case when A has real rank zero (with θ not being assumed to be an A-module map nor bounded). This result is also interesting by itself, and gives us a rough idea what kind of difficulties will come across without the A-linearity.

Proposition 3.5. *Let A be a unital C^*-algebra of real rank zero, and A_0 be the $*$-algebra generated by the idempotents in A. Suppose that there is an element $x_0 \in E$ such that $\langle x_0, x_0\rangle = 1$. If $\Psi : E \to F$ is a local map preserving orthogonality, then one can find $u \in Z(A)_+$ as well as an A_0-submodule $E_0 \subseteq E$ containing x_0 with $E_0^\perp = \{0\}$ such that*

$$\langle\Psi(x), \Psi(y)\rangle \ = \ u\langle x, y\rangle \quad (x, y \in E_0).$$

Proof. Define $u := \langle\Psi(x_0), \Psi(x_0)\rangle \in A_+$. Note that by Lemma 3.1, Ψ is an A_0-module map. Thus, $\Psi(xw) = \Psi(x)w$ for any symmetry $w \in A$ (as $w \in A_0$). Now, the argument of Proposition 2.2 tells us that $u \in Z(A)_+$. Let $z \in E$ such that $\langle x_0, z\rangle = 0$ and $\langle z, z\rangle \in A_0$. Then $z + x_0$ is also orthogonal to $z - x_0\langle z, z\rangle$. It follows from the orthogonality preserving property and the A_0-linearity of Ψ that

$$\langle\Psi(z), \Psi(z)\rangle \ = \ \langle\Psi(x_0), \Psi(x_0)\rangle\langle z, z\rangle \ = \ u\langle z, z\rangle.$$

Let $\mathcal{D} := \{D \subseteq E : x_0 \in D; \langle x, x\rangle \in A_0 \text{ and } \langle x, y\rangle = 0 \text{ for any } x \neq y \in D\}$. Take any maximal element M in $\mathcal{D}$, and define E_0 to be the linear spans of $x \cdot a$ ($x \in M; a \in A_0$). For any $y \in E_0$, we know that $\langle y, y\rangle, \langle x_0, y\rangle \in A_0$. Thus, $z = y - x_0\langle x_0, y\rangle$ is orthogonal to x_0 and $\langle z, z\rangle = \langle y, y\rangle - \langle y, x_0\rangle\langle x_0, y\rangle \in A_0$. Hence, the above implies that

$$\langle\Psi(y), \Psi(y)\rangle \ = \ \langle y, x_0\rangle\langle\Psi(x_0), \Psi(x_0)\rangle\langle x_0, y\rangle + \langle\Psi(z), \Psi(z)\rangle \ = \ u\langle y, y\rangle.$$

A polarization type argument tells us that $\langle\Psi(x),\Psi(y)\rangle = u\langle x,y\rangle$ $(x,y\in E_0)$. Suppose on the contrary that there exists $z\in E$ with $\|z\|=1$ and $\langle z,x\rangle=0$ for any $x\in E_0$. Let $a:=\langle z,z\rangle$ and $q_n:=e_a(\frac{1}{2^n},1]$ $(n\in\mathbb{N})$. There exist $d,b\in C^*(a)_+$ such that $q_5\leq ab\leq 1$, $d\leq a$, $dq_4=aq_4$ and $dq_5=d$. As $b^{1/2}d^{1/2}\leq 1$, $b^{1/2}d^{1/2}q_4=b^{1/2}a^{1/2}q_4=q_4$ and $b^{1/2}d^{1/2}q_5=b^{1/2}d^{1/2}$, we see that

$$\|z-zb^{1/2}d^{1/2}\|^2 \;=\; \|a-2ab^{1/2}d^{1/2}+abd\| \;\leq\; 2\|a(1-b^{1/2}d^{1/2})\| \;<\; 1/8.$$

Since $d^{1/2}\in \mathrm{her}(q_5)$, there exists $c\in A_0\cap\mathrm{her}(q_5)_+$ such that $\|d^{1/2}-c\|<1/8$ (because $\mathrm{her}(q_5)$ also has real rank zero; see e.g. [5]). If $z':=zb^{1/2}c$, then $\langle z',z'\rangle=cq_5abq_5c=c^2\in A_0$ (as $abq_5=q_5$) and $\|z-z'\|\leq\|z-zb^{1/2}d^{1/2}\|+\|zb^{1/2}d^{1/2}-zb^{1/2}c\|\leq 1/2$, which implies that $z'\neq 0$. Moreover, $\langle x,z'\rangle=\langle x,z\rangle b^{1/2}c=0$ for any $x\in M$, we see that $M\cup\{z'\}\in\mathcal{D}$, which is a contradiction. □

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