NONLINEAR ERGODIC THEOREM FOR POSITIVELY HOMOGENEOUS NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Recently, two retractions (projections) which are different from the metric projection and the sunny nonexpansive retraction in a Banach space were found. In this paper, using nonlinear analytic methods and new retractions, we prove a nonlinear ergodic theorem for positively homogeneous and nonexpansive mappings in a uniformly convex Banach space. The limit points are characterized by using new retractions.

1. Introduction

Let E be a real Banach space and let C be a nonempty subset of E. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

We denote by $F(T)$ the set of fixed points of T. In 1938, Yosida [28] proved the following strong convergence theorem for linear continuous operators in a Banach space.

Theorem 1.1 (Yosida [28]). *Let E be a Banach space and let T be a linear operator of E into itself. Suppose that there exists a constant C with $\|T^n\| \leq C$ for $n \in \mathbb{N}$ and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then, for each $x \in E$, the Cesàro means*

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converge strongly as $n \to \infty$ to $z \in F(T)$.

On the other hand, Baillon [2] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

Theorem 1.2 (Baillon [2]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to $z \in F(T)$.

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Bruck [5] extended Baillon's result to Banach spaces as follows:

Theorem 1.3 (Bruck [5]). *Let E be a uniformly convex Banach space whose norm is a Fréchet differentiable and let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to $z \in F(T)$.

However, the limit points $z \in F(T)$ in Theorems 1.1 and 1.3 are not characterized. Recently, two retractions (projections) which are different from the metric projection and the sunny nonexpansive retraction in a Banach space were found; see, for instance, Alber [1], and Ibaraki and Takahashi [11]. Such retractions are called the generalized projection and the sunny generalized nonexpansive retraction.

In this paper, using nonlinear analytic methods and new retractions which were found recently, we prove a nonlinear ergodic theorem for positively homogeneous and nonexpansive mappings in a uniformly convex Banach space. The limit points are characteralized by new retractions.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let A be a nonempty subset of E. We denote by $\overline{co}A$ the closure of the convex hull of A. The duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. Furthermore, we know from [23] that

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone, i.e.,

$$\langle x - y, Jx - Jy \rangle > 0, \quad \forall x, y \in E, \ x \neq y;$$

(v) if E has a Fréchet differentiable norm, then J is norm-to-norm continuous.

Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$. We also know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2; \tag{2.1}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle; \tag{2.2}$$

$$2\langle x-y, Jz-Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.3}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \ \text{ if and only if } \ x = y. \tag{2.4}$$

The following results were proved by Xu [27] and Kamimura and Takahashi [17].

Lemma 2.1 (Xu [27])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|ax + (1-a)y\|^2 \le a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)g(\|x-y\|)$$

for all $x, y \in B_r$ and $a \in [0,1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.2 (Kamimura and Takahashi [17])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \le \phi(x,y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.

Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. We know the following results.

Lemma 2.3 (Bruck [6])**.** *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E. let T be a nonexpansive mapping of C into itself. Define*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x, \quad \forall x \in C,\ n \in \mathbb{N}.$$

Then,

$$\lim_{n\to\infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.$$

Lemma 2.4 (Browder [4])**.** *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E. let T be a nonexpansive mapping of C into itself. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.*

Lemma 2.5 (Itoh and Takahashi [16])**.** *Let E be a strictly convex Banach space and let C be a nonempty, closed and convex subset of E. let T be a quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *generalized nonexpansive* [11] if $F(T) \ne \emptyset$ and

$$\phi(Tx, y) \le \phi(x, y), \quad \forall x \in C,\ y \in F(T).$$

Let E be a Banach space and let C be a closed and convex cone of E. A mapping $T : C \to C$ is called *positively homogeneous* if $T(\alpha x) = \alpha T(x)$ for all $x \in C$ and $\alpha \ge 0$.

Lemma 2.6 (Takahashi and Yao [26]). *Let E be a Banach space and let C be a closed and convex cone of E. Let $T : C \to C$ be a positively homogenuous nonexpansive mapping. Then, for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that*

$$\langle x - Tx, j \rangle \le 0,$$

where J is the duality mapping of E into E^*.

Using Lemma 2.6, Takahashi and Yao [26] proved the following result.

Lemma 2.7 (Takahashi and Yao [26]). *Let E be a smooth Banach space and let C be a closed and convex cone of E. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, T is a generalized nonexpansive mapping.*

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \ t \ge 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [10, 12, 11] for more details. The following results are in Ibaraki and Takahashi [11].

Lemma 2.8 (Ibaraki and Takahashi [11]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.9 (Ibaraki and Takahashi [11]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \le 0$ for all $y \in C$;*
(ii) *$\phi(Rx, z) + \phi(x, Rx) \le \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [18] also proved the following results:

Lemma 2.10 (Kohsaka and Takahashi [18]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.11 (Kohsaka and Takahashi [18]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Inthakon, Dhompongsa and Takahashi [15] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [13, 14].

Lemma 2.12 (Inthakon, Dhompongsa and Takahashi [15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.10 and 2.12.

Lemma 2.13 (Inthakon, Dhompongsa and Takahashi [15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [23, 24]. Using means and the Riesz theorem, we can obtain the following result; see [21] and [8, 9].

Lemma 2.14. *Let E be a reflexive Banach space, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y^* \rangle = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*.$$

Such a point z_0 in Lemma 2.14 is called the *mean vector* of $\{x_n\}$ for μ. This point z_0 plays a crucial role in this paper. The following result is in Hirano, Kido and Takahashi [8].

Lemma 2.15. *Let E be a uniformly convex Banach space and let C be a nonempty, closed and convex subset of E. Let T be a nonexpansive mapping of C into C such that $F(T) \neq \emptyset$. Let μ be a Banach limit on l^∞. Then the mean vector of $\{x_n\}$ for μ is in $F(T)$.*

The following result is in Lin, Takahashi and Yu [20].

Lemma 2.16 (Lin, Takahashi and Yu [20]). *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let D be a nonempty, closed and convex subset of E. Let $\{x_n\}$ be a bounded sequence in D and let μ be a mean on l^∞. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n \phi(x_n, z), \quad \forall z \in D,$$

then the mean vector z_0 of $\{x_n\}$ for μ is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

3. Lemmas

In the section, we first prove the following lemma which plays an important role for proving our main theorem.

Lemma 3.1. *Let E be a uniformly convex and smooth Banach space and let T be a positively homogeneous nonexpansive mapping of E into itself. Then for any $x \in C$, the sequence $\{T^n x\}$ is bounded and the set*

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$$

consists of one point z_0, where z_0 is a unique minimizer of $F(T)$ such that

$$\lim_{n\to\infty} \phi(T^n x, z_0) = \min\{\lim_{n\to\infty} \phi(T^n x, z) : z \in F(T)\}.$$

Proof. Since $T : E \to E$ is positively homogeneous and nonexpansive, it follows from Lemma 2.7 that T is generalized nonexpansive. Thus we have that for any $z \in F(T)$ and $x \in C$,

$$\phi(T^{n+1}x, z) \le \phi(T^n x, z) \le \cdots \le \phi(x, z), \quad \forall n \in \mathbb{N}.$$

Then $\{T^n x\}$ is bounded. Let μ be a Banach limit on l^∞. From Lemma 2.16, the mean vector $z_0 \in E$ of $\{T^n x\}$ for μ is a unique minimizer $z_0 \in E$ such that

$$\mu_n \phi(T^n x, z_0) = \min\{\mu_n \phi(T^n x, y) : \ y \in E\}.$$

We also know from Lemma 2.15 that $z_0 \in F(T)$. Furthermore, this $z_0 \in F(T)$ satisfies that

$$\mu_n \phi(T^n x, z_0) = \min\{\mu_n \phi(T^n x, y) : \ y \in F(T)\}.$$

Let us show that $z_0 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. If not, there exists some $k \in \mathbb{N}$ such that $z_0 \notin \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. By the separation theorem, there exists $y_0^* \in E^*$ such that

$$\langle z_0, y_0^* \rangle < \inf\left\{\langle z, y_0^* \rangle : z \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}\right\}.$$

Using the property of the Banach limit μ, we have that

$$\begin{aligned}
\langle z_0, y_0^* \rangle &< \inf\left\{\langle z, y_0^* \rangle : z \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}\right\} \\
&\le \inf\{\langle T^{k+n}x, y_0^* \rangle : n \in \mathbb{N}\} \\
&\le \mu_n \langle T^{k+n}x, y_0^* \rangle \\
&= \mu_n \langle T^n x, y_0^* \rangle \\
&= \langle z_0, y_0^* \rangle.
\end{aligned}$$

This is a contradiction. Thus we have that $z_0 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. Next we show that $\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$ consists of one point z_0. Assume that $z_1 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$. Since $z_1 \in F(T) = B(T)$, we have that

$$\phi(T^{n+1}x, z_1) \le \phi(T^n x, z_1), \quad \forall n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} \phi(T^n x, z_1)$ exists. Furthermore, we know from the property of a Banach limit μ that

$$\mu_n \phi(T^n x, z_1) = \lim_{n\to\infty} \phi(T^n x, z_1).$$

In general, since $\lim_{n\to\infty} \phi(T^n x, z)$ exists for every $z \in F(T)$, we define a function $g : F(T) \to \mathbb{R}$ as follows:

$$g(z) = \lim_{n\to\infty} \phi(T^n x, z), \quad \forall z \in F(T).$$

Since

$$\phi(z_0, z_1) = \phi(T^n x, z_1) - \phi(T^n x, z_0) - 2\langle T^n x - z_0, Jz_0 - Jz_1\rangle$$

for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\phi(z_0, z_1) &+ 2 \lim_{n\to\infty} \langle T^n x - z_0, Jz_0 - Jz_1\rangle \\ &= \lim_{n\to\infty} \phi(T^n x, z_1) - \lim_{n\to\infty} \phi(T^n x, z_0) \\ &\geq 0.\end{aligned}$$

Let $\epsilon > 0$. Then we have that

$$2 \lim_{n\to\infty} \langle T^n x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$2\langle T^n x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon$$

for every $n \in \mathbb{N}$ with $n \geq n_0$. Since $z_1 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$, we have

$$2\langle z_1 - z_0, Jz_0 - Jz_1\rangle \geq -\phi(z_0, z_1) - \epsilon.$$

We have from (2.3) that

$$\phi(z_1, z_1) + \phi(z_0, z_0) - \phi(z_1, z_0) - \phi(z_0, z_1) \geq -\phi(z_0, z_1) - \epsilon$$

and hence $\phi(z_1, z_0) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\phi(z_1, z_0) = 0$. Since E is strictly convex, we have $z_0 = z_1$. Therefore

$$\{z_0\} = \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T).$$

This completes the proof. □

For proving our main theorem (Theorem 4.1), we also need the following two lemmas.

Lemma 3.2. *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E. let T be a nonexpansive mapping of C into itself. For any $x \in S$, define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad \forall n \in \mathbb{N}.$$

If a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to a point u, then $u \in F(T)$.

Proof. We know from Lemma 2.3 that

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Since a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to a point u, we have from Lemma 2.4 that $u \in F(T)$. This completes the proof. □

Lemma 3.3. *Let E be a uniformly convex and smooth Banach space and let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $F(T)$. Furthermore, for any $x \in E$, $\lim_{n\to\infty} RT^n x$ exists in $F(T)$.*

Proof. We have from Lemma 2.5 that $F(T)$ is closed and convex. Furthermore, we have from Lemma 2.12 that $JF(T)$ are closed and convex. Then from Lemmas 2.8, 2.10 and 2.13 , there exists a unique sunny generalized nonexpansive retraction R of E onto $F(T)$. From Lemma 2.9, we know that

$$0 \leq \langle v - Rv, JRv - Ju\rangle, \quad \forall v \in C, \ u \in F(T). \tag{3.1}$$

We have from (3.1) and (2.3) that

$$\begin{aligned} 0 &\leq 2\langle v - Rv, JRv - Ju\rangle \\ &= \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv) - \phi(Rv,u). \end{aligned}$$

Hence we have that

$$\phi(Rv,u) \leq \phi(v,u) - \phi(v,Rv), \quad \forall v \in C, \ u \in F(T). \tag{3.2}$$

Since $\phi(Tz,u) \leq \phi(z,u)$ for any $u \in F(T)$ and $z \in C$, it follows from Lemma 2.11 that

$$\begin{aligned} \phi(T^n x, RT^n x) &\leq \phi(T^n x, RT^{n-1}x) \\ &\leq \phi(T^{n-1}x, RT^{n-1}x). \end{aligned}$$

Hence the sequence $\phi(T^n x, RT^n x)$ is nonincreasing. Putting $u = RT^n x$ and $v = T^m x$ with $n \leq m$ in (3.2), we have from Lemma 2.2 that

$$\begin{aligned} g(\|RT^m x - RT^n x\|) &\leq \phi(RT^m x, RT^n x) \\ &\leq \phi(T^m x, RT^n x) - \phi(T^m x, RT^m x) \\ &\leq \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x), \end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RT^n x\}$ is a Cauchy sequence. Therefore $\{RT^n x\}$ converges strongly to a point $q \in F(T)$. This completes the proof. □

4. Nonlinear Ergodic Theorem

Using Lemmas 3.1, 3.2 and 3.3, we now prove the following nonlinear ergodic theorem for positively homogeneous nonexpansive mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly convex and smooth Banach space. Let $T : E \to E$ be a positively homogeneous nonexpansive mapping. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in F(T)$. Additionally, if the norm of E is a Fréchet differentiable, then $z_0 = \lim_{n\to\infty} R_{F(T)} T^n x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. Let $x \in E$ and define $D = \{z \in E : \|z\| \leq \|x\|\}$. Then D is nonempty, bounded, closed and convex. Furthermore, since T is nonexpansive and $0 \in F(T)$, D is invariant under T and hence $\{T^n x\}$ and $\{S_n x\}$ are in D. We know from Lemma 3.1 that the set

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$$

consists of one point z_0. To prove that $\{S_n x\}$ converges weakly to z_0 in $F(T)$, it is sufficient to show that for any subsequence $\{S_{n_i}x\}$ of $\{S_n x\}$ such that $S_{n_i}x \rightharpoonup v$, $v \in F(T)$ and

$$v \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}.$$

From Lemma 3.2, we have that $v \in F(T)$. Next, we show that

$$v \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}.$$

Fix $k \in \mathbb{N}$. We have that for any $n_i \in \mathbb{N}$ with $n_i > k$,

$$S_{n_i}x = \frac{1}{n_i}(x + Tx + \cdots + T^k x) + \frac{n_i - (k+1)}{n_i} \cdot \frac{1}{n_i - (k+1)}(T^{k+1}x + \cdots + T^{n_i - 1}).$$

Thus from $S_{n_i}x \rightharpoonup v$, we have

$$\frac{1}{n_i - (k+1)}(T^{k+1}x + \cdots + T^{n_i-1}) \rightharpoonup v$$

and hence $v \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. Since $k \in \mathbb{N}$ is arbitrary, we have that

$$v \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}.$$

Therefore $\{S_n x\}$ converges weakly to a point z_0 of $F(T)$.

Additionally, assume that the norm of E is a Fréchet differentiable. We have from Lemma 3.3 that there exists the sunny generalized nonexpansive retraction $R = R_{F(T)}$ of E onto $F(T)$ and $\{RT^n x\}$ converges strongly to a point $q \in F(T)$. Rewriting the characterization of the retraction R, we have that

$$0 \le \left\langle T^k x - RT^k x, JRT^k x - Ju \right\rangle, \quad \forall u \in F(T)$$

and hence

$$\begin{aligned}\left\langle T^k x - RT^k x, Ju - Jq \right\rangle &\le \left\langle T^k x - RT^k x, JRT^k x - Jq \right\rangle \\ &\le \|T^k x - RT^k x\| \cdot \|JRT^k x - Jq\| \\ &\le K\|JRT^k x - Jq\|,\end{aligned}$$

where K is an upper bound for $\|T^k x - RT^k x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$ and deviding by n, we arrive to

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} RT^k x, Ju - Jq \right\rangle \le \frac{K}{n}\sum_{k=0}^{n-1}\|JRT^k x - Jq\|.$$

Letting $n \to \infty$ and remembering that J is continuous because the norm of E is a Fréchet differentiable, we get that

$$\langle z_0 - q, Ju - Jq \rangle \le 0.$$

This holds for any $u \in F(T)$. Putting $u = z_0$, we have $\langle z_0 - q, Jz_0 - Jq \rangle \le 0$. Since J is monotone, we have $\langle z_0 - q, Jz_0 - Jq \rangle = 0$. Since E is strictly convex, we have $z_0 = q$. Thus $z_0 = \lim_{n\to\infty} R_{F(T)}T^n x$. □

Compare Theorem 4.1 with Theorem 1.3. Though the assumption of a mapping in Theorem 4.1 is stronger than that of Theorem 1.3, the assumption of a Banach space is weaker. Furthermore, the limit points are characterized by sunny generalized nonexpansive retractions.

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