# KAPLANSKY THEOREM FOR COMPLETELY REGULAR SPACES

LEI LI AND NGAI-CHING WONG

Abstract. Let $X, Y$ be realcompact spaces or completely regular spaces consisting of $G_\delta$-points. Let $\phi$ be a linear bijective map from $C(X)$ (resp. $C^b(X)$) onto $C(Y)$ (resp. $C^b(Y)$). We show that if $\phi$ preserves nonvanishing functions, that is,

$$f(x) \neq 0, \forall\, x \in X, \quad \Longleftrightarrow \quad \phi(f)(y) \neq 0, \forall\, y \in Y,$$

then $\phi$ is a weighted composition operator

$$\phi(f) = \phi(1) \cdot f \circ \tau,$$

arising from a homeomorphism $\tau : Y \to X$. This result is applied also to other nice function spaces, e.g., uniformly or Lipschitz continuous functions on metric spaces.

## 1. Introduction

The problem here is how to recover a topological space $X$ from the set $C(X)$ (resp. $C^b(X)$) of continuous (resp. bounded continuous) (real- or complex-valued) functions on $X$. We say that a net $\{x_\lambda\} \subset X$ converges to $x$ in the *weak topology* $\sigma(X, C(X))$ if $f(x_\lambda) \to f(x)$ for all $f$ in $C(X)$. It is easy to see that the weak topology $\sigma(X, C^b(X))$ coincides with $\sigma(X, C(X))$. A well-known fact states that $X$ carries the weak topology $\sigma(X, C(X))$ if and only if $X$ is completely regular (see, e.g., [9, Theorem 3.6]). In this sense, a completely regular topological space is determined by all its continuous functions.

Assume $X$ is completely regular throughout this paper. The set $C(X)$ and $C^b(X)$ carry the natural algebraic, lattice, and Banach space (for $C^b(X)$), structures. It is plausible that the algebra, the vector lattice, or the Banach space structures of $C(X)$ or $C^b(X)$ can also determine the topology of $X$.

*Question* 1.1. Suppose that there is an algebra (or lattice, or isometrically linear) isomorphism $\phi : C(X) \to C(Y)$ or $\phi : C^b(X) \to C^b(Y)$, can we conclude that the completely regular spaces $X$ and $Y$ are homeomorphic?

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In the literature, there are several well-known results in this line. For example, every ring isomorphism $\phi : C(X) \to C(Y)$ (resp. $\phi : C^b(X) \to C^b(Y)$) gives rise to a homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ (resp. $\tau^\beta : \beta Y \to \beta X$) between the Hewitt-Nachbin realcompactifications $\upsilon X$ and $\upsilon Y$ (resp. Stone-Čech compactifications $\beta X$ and $\beta Y$) of the completely regular spaces $X$ and $Y$, respectively. However, $X$ and $Y$ might be non-homeomorphic in both cases, unless they are both realcompact or compact to start with (see Example 1.2 below).

Let us sketch a proof here. Recall that every $f$ in $C(X)$ gives rise to a *zero set*

$$z(f) = \{x \in X : f(x) = 0\},$$

and denote by

$$Z(\mathcal{A}(X)) = \{z(f) : f \in \mathcal{A}(X)\}$$

for any subset $\mathcal{A}(X)$ of $C(X)$. In particular, $Z(C(X)) = Z(C^b(X))$, and denote it by $Z(X)$ for simplicity. A *z-filter* $\mathcal{F}$ on $X$ is a filter of zero sets in $Z(X)$. Call $\mathcal{F}$ a *z-ultrafilter* if it is a maximal $z$-filter; and call $\mathcal{F}$ *prime* if $A \in \mathcal{F}$ or $B \in \mathcal{F}$ whenever $X = A \cup B$ and $A, B \in Z(X)$. Associated to each $z$-ultrafilter $\mathcal{F}$ a maximal ideal $I$ of $C(X)$ consisting of all continuous functions $f$ such that $z(f) \in \mathcal{F}$. Call $\mathcal{F}$ *fixed* if $\bigcap \mathcal{F}$ is a singleton, and call $\mathcal{F}$ *real* if the quotient field $C(X)/I$ is isomorphic to $\mathbb{R}$ (assuming the underlying field is $\mathbb{R}$). The Stone-Čech compactification $\beta X$ can be identified with the set of *all* $z$-ultrafilters on $X$. In this setting, $X$ consists of all fixed $z$-ultrafilters. The Hewitt-Nachbin realcompactification $\upsilon X$ consists of all *real* $z$-ultrafiliters. Clearly, $X$ is compact if and only if $X = \beta X$. Call $X$ a *realcompact* space if $X = \upsilon X$. In fact, $X$ is realcompact if and only if every prime $z$-filter with the countable intersection property is fixed. For instance, the Linderlöf (and thus separable metric) spaces are realcompact, and discrete spaces of non-measurable cardinality are another examples. Especially, all subspaces of the Euclidean spaces $\mathbb{R}^n$ (and $\mathbb{C}^n$ as well) are realcompact. In general, $X$ is realcompact if and only if $X$ is homeomorphic to a closed subspace of a product of real lines. However, the order interval $[0, \omega_1)$ is not realcompact, where $\omega_1$ is the first uncountable ordinal. As ring isomorphisms preserve $z$-ultrafilters and real $z$-ultrafilters, the above results follow. We refer to the books [9] and [18] for more information about $z$-ultrafilters and realcompact spaces.

On the other hand, the classical Banach-Stone theorem tells us that the geometric structure of the Banach space $C^b(X)$ determines the topology of its Stone-Čech compactification $\beta X$. In the special case when $X, Y$ are compact, if $\phi : C(X) \to C(Y)$ is a surjective linear isometry then there is a homeomorphism $\tau : Y \to X$ and a unimodular continuous weight function $h$ in $C(Y)$ such that $\phi$ is the weighted composition operator $\phi(f) = h \cdot f \circ \tau$. In general, when $X, Y$ are completely regular spaces, since $C^b(X) \cong C(\beta X)$ and $C^b(Y) \cong C(\beta Y)$ as Banach spaces, there exists a surjective linear isometry between $C^b(X)$ and $C^b(Y)$ if and only if $\beta X$ and $\beta Y$ are homeomorphic (see, e.g., [9]).

When $X, Y$ are compact Hausdorff spaces, Kaplansky obtained in [14] yet another criterion: every lattice isomorphism $\phi : C(X) \to C(Y)$ also gives rise to a homeomorphism $\tau : Y \to X$; and he also showed in [15] that if $\phi$ is, in addition, additive then $\phi(f) = h \cdot f \circ \tau$ with a strictly positive weight function $h$ in $C(Y)$. Moreover, he showed that a positive linear map $\phi : C(X) \to C(Y)$ is a lattice isomorphism if and only if $\phi$ *preserves nonvanishing functions* (in two directions), that is,

$$z(f) = \emptyset \quad \Leftrightarrow \quad z(\phi(f)) = \emptyset, \quad \forall\, f \in C(X).$$

This starts a popular research subject of studying invertibility or spectrum preserving linear maps of Banach algebras (see, e.g., [4, 5]).

Nevertheless, the following example tells us that the algebraic, geometic and lattice structures of the Banach algebra $C^b(X)$ altogether are still not enough to determine the topology of a realcompact space.

**Example 1.2** (see [9, 4M])**.** Let $\Sigma$ be $\mathbb{N} \cup \{\sigma\}$ (where $\sigma \in \beta\mathbb{N}\backslash\mathbb{N}$). Clearly, $\mathbb{N}$ is dense in $\Sigma$, and every function $f$ in $C^b(\mathbb{N})$ can be extended uniquely to a function $f^\sigma$ in $C^b(\Sigma)$. Although the bijective linear map $\phi$ from $C^b(\mathbb{N})$ onto $C^b(\Sigma)$ defined by $f \mapsto f^\sigma$ provides an isometric, algebraic and lattice isomorphism, the realcompact spaces $\mathbb{N}$ and $\Sigma$ are not homeomorphic.

Notice that the map $\phi$ in Example 1.2 does not preserve nonvanishing functions. In Theorems 2.2 and 2.9 below, we will show that every bijective linear nonvanishing preserver between some nice subspaces of continuous functions is a weighted composition operator $f \mapsto h \cdot f \circ \tau$ arising from a homeomorphism $\tau$ between the realcompactifications of the underlying completely regular spaces. This in particular tells us that the property of a linear map preserving nonvanishing functions is stronger than those being multiplicative, lattice isomorphic, and isometric, and thus supplements many results in literatures, e.g., [1, 2, 7, 11, 12, 17].

## 2. Main Results

The underlying scalar field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, and we will assume that $\mathcal{A}(X)$ is a vector sublattice (self-adjoint if $\mathbb{K} = \mathbb{C}$) of $C(X)$ containing all constant functions in the following. Denote by $\mathcal{A}^b(X) := \mathcal{A}(X) \cap C^b(X)$ the vector sublattice of $\mathcal{A}(X)$ consisting of bounded functions, and by $\mathcal{A}(X)_+$ the subset of $\mathcal{A}(X)$ consisting of non-negative real-valued functions. For any $f$ in $\mathcal{A}(X)$, we can decompose $f = f_1 - f_2 + i(f_3 - f_4)$ in a unique way such that $f_1, f_2, f_3, f_4 \in \mathcal{A}(X)_+$ and $f_1 f_2 = f_3 f_4 = 0$. Write $|f| := f_1 + f_2 + f_3 + f_4$. Clearly, $|f| \geq 0$ and $z(|f|) = z(f)$.

**Definition 2.1.** We say that a subspace $\mathcal{A}(X)$ of $C(X)$ is

(1) *completely regular* if for every point $x$ and closed subset $F$ of $X$ with $x \notin F$, there is an $f$ in $\mathcal{A}(X)$ such that $x_0 \notin z(f)$ and $F \subseteq z(f)$;
(2) *full* if $Z(\mathcal{A}(X)) = Z(X)$;

(3) *nice* if for any sequence $\{f_n\}$ in $\mathcal{A}^b(X)_+$, there exists a sequence of strictly positive numbers $\{\lambda_n\}$ such that $\sum_{n=1}^{\infty} \lambda_n f_n$ converges pointwisely to a function $f$ in $\mathcal{A}(X)$.

Note that a full subspace of $C(X)$ is completely regular, but might not be normal, i.e., separating disjoint closed sets. For instance, the space $\mathrm{Lip}(X)$ of all Lipschitz continuous functions on the metric space $X = (-1, 0) \cup (0, 1)$ is full but not normal.

The following Kaplansky type theorem can be considered as a generalization of the Gleason-Kahane-Zelazko Theorem [10, 13].

**Theorem 2.2.** *Suppose that $X$ and $Y$ are realcompact spaces. Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be vector sublattices of $C(X)$ and $C(Y)$ containing all constant functions, respectively. Assume $\mathcal{A}(X)$ is nice and completely regular, and $\mathcal{A}(Y)$ is full. Let $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ be a bijective linear map preserving nonvanishing functions. Then there is a dense subset $Y_1$ of $Y$, containing all $G_\delta$ points in $Y$, and a homeomorphism $\tau : Y_1 \to X$ such that*

$$\phi(f)(y) = \phi(1)(y) f(\tau(y)), \quad \forall\, f \in \mathcal{A}(X), \forall\, y \in Y_1. \tag{2.1}$$

*In case all points of $Y$ are $G_\delta$, or in case $\mathcal{A}(X)$ is full and $\mathcal{A}(Y)$ is nice, we have $Y_1 = Y$.*

We will establish the proof of Theorem 2.2 in several lemmas.

**Lemma 2.3.** *$\phi$ is biseparating, i.e.,*

$$fg = 0 \text{ on } X \quad \Leftrightarrow \quad \phi(f)\phi(g) = 0 \text{ on } Y.$$

*Proof.* Suppose that $f$ and $g$ belong to $\mathcal{A}(X)$ with $fg = 0$, but $\phi(f)\phi(g) \neq 0$. Without loss of generality, we can assume that there exists a $y_0$ in $Y$ such that $\phi(f)(y_0) = \phi(g)(y_0) = 1$.

Define $h$ in $\mathcal{A}(Y)$ by

$$h(y) = \max\left\{0, \frac{1}{2} - \operatorname{Re}\phi(f)(y), \frac{1}{2} - \operatorname{Re}\phi(g)(y)\right\}, \quad \forall\, y \in Y;$$

and put

$$k = \phi^{-1}(h).$$

**Claim**: $z(\phi(f) + \phi(k)) = \emptyset$.

Indeed, assume on the contrary that $y$ belongs to $z(\phi(f) + \phi(k))$, that is,

$$\phi(f)(y) + \phi(k)(y) = \phi(f)(y) + h(y) = 0.$$

This provides a contradiction

$$h(y) \geq \frac{1}{2} - \operatorname{Re}\phi(f)(y) = \frac{1}{2} + h(y).$$

It follows from $z(\phi(f)+\phi(k))=\emptyset$ that $z(f+k)=\emptyset$. In a similar way, we also have $z(g+k)=\emptyset$. Notice that $z(f)\cap z(k)\subseteq z(f+k)$ and $z(g)\cap z(k)\subseteq z(g+k)$. We thus have $z(f)\cap z(k)=z(g)\cap z(k)=\emptyset$. By the assumption $z(f)\cup z(g)=X$, one can conclude $z(k)=\emptyset$. This is a contradiction since $(\phi k)(y_0)=h(y_0)=0$ and $\phi$ is nonvanishing preserving. Hence, $\phi(f)\phi(g)=0$, as asserted.

Similarly, we can derive that $\phi^{-1}$ is also separating, and hence $\phi$ is a biseparating map. □

We note that a biseparating mapping might not be nonvanishing preserving as shown in Example 1.2. The following lemma is motivated by the results in [6, 17].

**Lemma 2.4.** *$\phi$ sends functions without common zeros to functions without common zeros. That is, for any $m$ in $\mathbb{N}$ and $f_1,\ldots,f_m$ in $\mathcal{A}(X)$, we have*

$$\bigcap_{k=1}^{m} z(f_k)=\emptyset \quad\iff\quad \bigcap_{k=1}^{m} z(\phi(f_k))=\emptyset.$$

*Proof.* Note first that $\phi(1)$ is nonvanishing on $Y$. Define $\psi(f):=\phi(f)/\phi(1)^{-1}$. It is easy to see that $\psi$ is an injective linear map from $\mathcal{A}(X)$ into $C(Y)$, and $z(\psi(f))=z(\phi(f))$ for all $f$ in $\mathcal{A}(X)$.

**Claim**. $\psi$ sends non-negative real functions to non-negative real functions.

Let $f\geq 0$ be in $\mathcal{A}(X)$, that is, $f(x)\geq 0$ for all $x$ in $X$, and let $\lambda$ be a non-positive scalar in $\mathbb{K}\setminus[0,+\infty)$. As $f-\lambda$ is nonvanishing on $X$, we can see that $\phi(f)-\lambda\phi(1)$ is nonvanishing on $Y$. Therefore, $\psi(f)-\lambda$ is also nonvanishing on $Y$. Since $\lambda$ is an arbitrary non-positive real number, we see that $\psi(f)$ assumes values from $[0,+\infty)$.

Inherited from $\phi$, the new map $\psi$ is also biseparating. It follows that $\psi(|f|)=|\psi(f)|$ for all $f$ in $\mathcal{A}(X)$. Now, suppose that $f_1,\ldots,f_m$ belong to $\mathcal{A}(X)$ with

$$\emptyset=\bigcap_{i=1}^{m} z(f_i)=\bigcap_{i=1}^{m} z(|f_i|)=z(\sum_{i=1}^{m}|f_i|).$$

Observe that

$$\begin{aligned}\bigcap_{k=1}^{m} z(\phi(f_k)) &= \bigcap_{k=1}^{m} z(\psi(f_k)) = \bigcap_{k=1}^{m} z(|\psi(f_k)|)\\ &= \bigcap_{k=1}^{m} z(\psi(|f_k|)) = z(\sum_{k=1}^{m}\psi(|f_k|))\\ &= z(\psi(\sum_{k=1}^{m}|f_k|)) = z(\phi(\sum_{k=1}^{m}|f_k|))=\emptyset.\end{aligned}$$

The proof for the other direction is similar. □

**Lemma 2.5.** *$\phi$ preserves zero-set containments, i.e.,*

$$z(f)\subseteq z(g) \quad\iff\quad z(\phi(f))\subseteq z(\phi(g)),\quad \forall\, f,g\in\mathcal{A}(X).$$

*Proof.* Assume $z(f) \subseteq z(g)$. Let $y$ in $Y$ be such that $\phi(g)(y) \neq 0$. As in the proof of Lemma 2.3, we can find a function $k$ in $\mathcal{A}(X)$ such that

$$z(\phi(g) + \phi(k)) = \emptyset \quad \text{and} \quad \phi(k)(y) = 0.$$

By the assumption,

$$z(f) \cap z(k) \subseteq z(g) \cap z(k) \subseteq z(g + k) = \emptyset.$$

It follows from Lemma 2.4 that

$$z(\phi(f)) \cap z(\phi(k)) = \emptyset.$$

In particular, $\phi(f)(y) \neq 0$, as asserted. The other direction is similar. □

For any $x_0$ in $X$, let

$$\mathcal{K}_{x_0} = \{f \in \mathcal{A}(X) : f(x_0) = 0\},$$

and

$$\mathcal{Z}_{x_0} = Z(\phi(\mathcal{K}_{x_0})) = \{z(\phi f) : f \in \mathcal{K}_{x_0}\}.$$

**Lemma 2.6.** *$\mathcal{Z}_{x_0}$ is a prime $z$-filter on $Y$ with the countable intersection property.*

*Proof.* We first note that by the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$, every zero set $A$ in $Z(Y)$ can be written as $A = z(\phi(f))$ for some $f$ in $\mathcal{A}(X)$.

Because $\phi$ is nonvanishing preserving, the empty set is not in $\mathcal{Z}_{x_0}$. Let $f \in \mathcal{K}_{x_0}$ and $C = z(\phi(g)) \in Z(Y)$ such that $z(\phi(f)) \subseteq C$. Then $z(f) \subseteq z(g)$ since $\phi$ preserves zero-set containments by Lemma 2.5, and hence $g \in \mathcal{K}_{x_0}$. This means that $C \in \mathcal{Z}_{x_0}$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{K}_{x_0}$. Set $g_n = \min\{1, |f_n|\}$ in $\mathcal{A}^b(X)$, Clearly, $z(g_n) = z(f_n)$. Since $\mathcal{A}(X)$ is nice, we can find a strictly positive sequence $\{\lambda_n\}$ such that the pointwise limit $g_0 = \sum_{n=1}^{\infty} \lambda_n g_n$ is in $\mathcal{A}(X)$. Obviously,

$$x_0 \in z(g_0) = \bigcap_{n=1}^{\infty} z(g_n) = \bigcap_{n=1}^{\infty} z(f_n).$$

It follows from Lemma 2.5 that

$$\emptyset \neq z(\phi g_0) \subseteq \bigcap_{n=1}^{\infty} z(\phi(f_n)).$$

This establishes that $\mathcal{Z}_{x_0}$ is a $z$-filter with the countable intersection property.

Finally, we check the primeness of the $z$-filter $\mathcal{Z}_{x_0}$. Let $f, g$ in $\mathcal{A}(X)$ be such that $z(\phi f) \cup z(\phi g) = Y$. Then $z(f) \cup z(g) = X$ since $\phi$ is biseparating by Lemma 2.3. As a result, $x_0$ must be in $z(f)$ or $z(g)$. This means that $f$ or $g$ belongs to $\mathcal{K}_{x_0}$, and thus proves $\mathcal{Z}_{x_0}$ is prime. □

Since $Y$ is realcompact, by Lemma 2.6 we see that the intersection of $\mathcal{Z}_{x_0}$ is a singleton, and denote it by $\{\sigma(x_0)\}$. In other words,

$$f(x_0) = 0 \quad \Longrightarrow \quad \phi(f)(\sigma(x_0)) = 0, \quad \forall f \in \mathcal{A}(X).$$

**Lemma 2.7.** *For any $f$ in $\mathcal{A}(X)$, we have*

$$(\phi f)(\sigma(x)) = (\phi 1)(\sigma(x)) f(x), \quad \forall x \in X. \tag{2.2}$$

*Proof.* For any $f$ in $\mathcal{A}(X)$ and $x$ in $X$, the function $f - f(x)$ is in $\mathcal{K}_x$. It follows

$$\phi(f - f(x))(\sigma(x)) = 0,$$

and thus $(\phi f)(\sigma(x)) = \phi(1)(\sigma(x)) \cdot f(x)$. □

*Proof of Theorem 2.2.* Firstly, we shall see that $\sigma : X \to Y$ is one-to-one. Suppose that $x \neq x' \in X$ and $\sigma(x) = \sigma(x')$. Choose a function $f$ from $\mathcal{A}(X)$ such that $f(x) = 0$ and $f(x') \neq 0$. By (2.2), we have the following contradiction. Note that $\phi 1$ is non-vanishing.

$$(\phi f)(\sigma(x)) = (\phi 1)(\sigma(x)) f(x) = 0$$

and

$$(\phi f)(\sigma(x')) = (\phi 1)(\sigma(x')) f(x') \neq 0.$$

Secondly, we claim that $\sigma(X)$ is dense in $Y$. Indeed, if there exists a $y$ in $Y \setminus \overline{\sigma(X)}$, then we can choose a function $f_1$ from $\mathcal{A}(X)$ such that $(\phi f_1)(y) = 1$ and $\phi(f_1) \equiv 0$ on $\sigma(X)$ by the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$. For any $x$ in $X$, we have

$$(\phi f_1)(\sigma(x)) = (\phi 1)(\sigma(x)) f_1(x) = 0.$$

This forces $f_1 = 0$. In turn, $(\phi f_1)(y) = 0$, which is impossible.

Thirdly, $\sigma$ induces a homeomorphism from $X$ onto $\sigma(X)$. Suppose on the contrary that a net $\{x_\lambda\}$ converges to $x_0$ in $X$ but $\{\sigma(x_\lambda)\}$ does not converge to $\sigma(x_0)$ in $Y$. Without loss of generality, we can assume that all $\sigma(x_\lambda)$ lie outside an open neighborhood of $\sigma(x)$. Find a function $g$ in $\mathcal{A}(X)$ such that $(\phi g)(\sigma(x_\lambda)) = 0$ for all $\lambda$ and $(\phi g)(\sigma(x_0)) \neq 0$. Since

$$0 = (\phi g)(\sigma(x_\lambda)) = (\phi 1)(\sigma(x_\lambda)) g(x_\lambda)$$

and $\phi 1$ is nonvanishing, $g(x_\lambda) = 0$ for all $\lambda$ and hence $g(x_0) = 0$. This forces

$$(\phi g)(\sigma(x_0)) = (\phi 1)(\sigma(x_0)) g(x_0) = 0.$$

This is a contradiction. Similarly, we can prove that $\sigma^{-1}$ is continuous from $\sigma(X)$ into $X$. Setting $Y_1 = \sigma(X)$ and $\tau = \sigma^{-1} : \sigma(X) \to X$, we get the desired assertion (2.1).

Now we verify that $Y_1$ contains all $G_\delta$ points in $Y$. Suppose $y$ in $Y \setminus Y_1$ is a $G_\delta$ point. It follows from the fullness of $\mathcal{A}(Y) = \phi(\mathcal{A}(X))$ that there is an $f$ in $\mathcal{A}(X)$ such that $z(\phi(f)) = \{y\}$. In particular, $\phi(f)$ is nonvanishing on $Y_1$. Then, the representation (2.2) ensures that $z(f) = \emptyset$. This contradicts to the non-vanishing preserving property of $\phi$. Hence, $y \in Y_1$. In the case $Y$ consists of $G_\delta$ points, $Y = Y_1$.

Lastly, we show that $\sigma : X \to Y$ is surjective when $\mathcal{A}(X)$ is full and $\mathcal{A}(Y)$ is nice. In this case, we have $Z(\mathcal{A}(X)) = Z(X)$. For any $y_0$ in $Y$, set

$$\mathcal{Z}_{y_0} = \{z(f) : (\phi f)(y_0) = 0\}.$$

Arguing as in Lemma 2.6, we see that $\mathcal{Z}_{y_0}$ is also a prime $z$-filter on $X$ with the countable intersection property. Since $X$ is realcompact, $\bigcap \mathcal{Z}_{y_0}$ is a singleton and denoted it by $\{x_0\}$. It is then easy to see that $\sigma(x_0) = y_0$. □

*Remark* 2.8. (1) If $\mathcal{A}(X)$ is a uniformly closed unital subalgebra of $C^b(X)$, then $\mathcal{A}(X)$ is a nice sublattice. See, e.g., [9, Lemma 16.2].
(2) When $\mathcal{A}(X) \subseteq C(X)$ and $\mathcal{A}(Y) \subseteq C(Y)$ are endowed with the compact-open topology, or $\mathcal{A}(X) \subseteq C^b(X)$ and $\mathcal{A}(Y) \subseteq C^b(Y)$ endowed with the uniform topology, $\phi$ is automatically continuous. A proof for these facts make use of the weighted composition representation (2.1) and is left to the readers.

Note that every continuous map $\psi : X \to Y$ between completely regular spaces can be lifted uniquely to a continuous map $\psi^\upsilon : \upsilon X \to \upsilon Y$ between their realcompactifications. In particular, every $f$ in $C(X)$ can be lifted uniquely to an $f^\upsilon$ in $C(\upsilon X)$ with the same range $f^\upsilon(\upsilon X) = f(X)$ (see, e.g., [9, Theorem 8.7 and 8B]). Consequently, $f$ is nonvanishing if and only if $f^\upsilon$ is nonvanishing.

**Theorem 2.9.** *Suppose that $X, Y$ are completely regular spaces with realcompactifications $\upsilon X$, $\upsilon Y$, respectively. Let $\mathcal{A}(X), \mathcal{A}(Y)$ be nice and full vector sublattices of $C(X), C(Y)$ containing constant functions, respectively. Assume that $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ is a bijective linear nonvanshing preserver. Then, there exists a homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ such that*

$$(\phi f)^\upsilon(y) = (\phi 1)^\upsilon(y) f^\upsilon(\tau^\upsilon(y)), \quad \forall\, f \in \mathcal{A}(X), y \in \upsilon Y.$$

*In case both $X$ and $Y$ consist of $G_\delta$-points, $\tau^\upsilon$ restricts to a homeomorphism $\tau : Y \to X$ such that*

$$\phi(f)(y) = \phi(1)(y) f(\tau(y)), \quad \forall f \in \mathcal{A}(X), y \in Y.$$

*Proof.* Denote by $\mathcal{A}(\upsilon X)$ the nice and full vector sublattice of $C(\upsilon X)$ consisting of the unique extensions $f^\upsilon : \upsilon X \to \mathbb{K}$ of all $f$ in $\mathcal{A}(X)$. Since $\phi : \mathcal{A}(X) \to \mathcal{A}(Y)$ is nonvanishing preserving, $\phi^\upsilon : \mathcal{A}(\upsilon X) \to \mathcal{A}(\upsilon Y)$ defined by $\phi^\upsilon(f^\upsilon) = (\phi f)^\upsilon$ is also nonvanishing preserving. By Theorem 2.2, there is a homeomorphism $\tau^\upsilon : \upsilon Y \to \upsilon X$ such that

$$(\phi^\upsilon f^\upsilon)(y) = (\phi^\upsilon 1^\upsilon)(y) f^\upsilon(\tau^\upsilon(y)), \quad \forall\, f^\upsilon \in \mathcal{A}(\upsilon X), y \in \upsilon Y.$$

Finally, since $\upsilon X \backslash X$ and $\upsilon Y \backslash Y$ contain no $G_\delta$-points (see, e.g., [9, p. 132]), $\tau^\upsilon(Y) = X$ when both $X, Y$ consists of $G_\delta$-points. □

Recall that a metric space $(X, d)$ is said to be *quasi-convex* if there is a constant $C > 0$ such that for any points $x, y$ in $X$ there is a continuous curve joining $x$ to

$y$ in $X$ with length not greater than $Cd(x, y)$ (see [8]). The following corollary demonstrates the applicability of our main results. We do not claim the full originality, and some content can be seen in other papers, e.g., [3] for Part (c) in the case $X, Y$ are complete metric spaces.

**Corollary 2.10.** Suppose $\phi$ is a bijective linear nonvanishing preserver between the following function spaces. Then there is a homeomorphism $\tau : Y \to X$ such that

$$\phi(f)(y) = \phi(1)(y)f(\tau(y)), \quad \forall y \in Y. \tag{2.3}$$

(a) $\phi : C(X) \to C(Y)$ or $\phi : C^b(X) \to C^b(Y)$, where $X, Y$ are both realcompact spaces, or are both completely regular spaces such that all points of $X, Y$ are $G_\delta$-points.
(b) $\phi : UC(X) \to UC(Y)$ or $\phi : UC^b(X) \to UC^b(Y)$, where $UC(X), UC(Y)$ consist of uniformly continuous functions on the metric spaces $X, Y$, respectively. In this case, $\tau$ is a uniform homeomorphsim from $Y$ onto $X$.
(c) $\phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ or $\phi : \mathrm{Lip}^b(X) \to \mathrm{Lip}^b(Y)$, where $\mathrm{Lip}(X), \mathrm{Lip}(Y)$ consist of Lipschitz continuous functions on the metric spaces $X, Y$, respectively. In the case $\phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$, $\tau$ is a Lipschitz homeomorphism from $Y$ onto $X$. We get the same conclusion in the other case, provided that $X, Y$ are quasi-convex.

*Proof.* Note that all function spaces here are full and nice, and closed in the lattice operations. So Theorems 2.2 and 2.9 apply.

For (b), it follows from (2.3) that $\phi(1)(y)\phi^{-1}(1)(\tau(y)) = 1$ for all $y$ in $Y$. Define a linear map $\psi(f) = \phi(\phi^{-1}(1)f) = f \circ \tau$ from $UC^b(X)$ into $UC(Y)$. Using the arguments in [16, Theorem 2.3], we can show that $\tau$ is uniformly continuous. Similarly, $\tau^{-1}$ is also uniformly continuous.

In a similar manner, the assertion (c) follows from [8, Theorems 3.9 and 3.12]. □

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(Lei Li) School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, China

*E-mail address*: leilee@nankai.edu.cn

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

*E-mail address*: wong@math.nsysu.edu.tw