ON C*-ALGEBRAS CUT DOWN BY CLOSED PROJECTIONS: CHARACTERIZING ELEMENTS VIA THE EXTREME BOUNDARY

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Abstract. Let A be a C*-algebra. Let z be the maximal atomic projection and p a closed projection in A^{**}. It is known that x in A^{**} has a continuous atomic part, *i.e.* $zx = za$ for some a in A, whenever x is uniformly continuous on the set of pure states of A. Under some additional conditions, we shall show that if x is uniformly continuous on the set of pure states of A supported by p, or its weak* closure, then pxp has a continuous atomic part, *i.e.* $zpxp = zpap$ for some a in A.

1. Introduction

Let A be a C*-algebra with Banach dual A^* and double dual A^{**}. Let

$$Q(A) = \{\varphi \in A^* : \varphi \geq 0 \text{ and } \|\varphi\| \leq 1\}$$

be the quasi-state space of A. When $A = C_0(X)$ for some locally compact Hausdorff space X, the weak* compact convex set $Q(C_0(X))$ consists of all positive regular Borel measures μ on X with $\|\mu\| = \mu(X) \leq 1$. In this case, the extreme boundary of $Q(C_0(X)) \cong X \cup \{\infty\}$. The point ∞ at infinity is isolated if and only if X is compact. For a non-abelian C*-algebra A, the extreme boundary of $Q(A)$ is the pure state space $P(A) \cup \{0\}$, in which $P(A)$ consists of pure states of A and the zero functional 0 is isolated if and only if A is unital. In the Kadison function representation (see *e.g.* [16]), the self-adjoint part A^{**}_{sa} of the W*-algebra A^{**} is isometrically and order isomorphic to the ordered Banach space of all bounded affine real-valued functionals on $Q(A)$ vanishing at 0. Moreover, x is in A_{sa} if and only if in addition x is weak* continuous on $Q(A)$.

Let z be the maximal atomic projection in A^{**}. Note that $A^{**} = (1-z)A^{**} \oplus zA^{**}$; in which zA^{**} is the direct sum of type I factors and $(1-z)A^{**}$ has no type I factor direct summand of A^{**}. In particular, z is a central projection in A^{**} supporting all pure states of A. In other words, $\varphi(x) = \varphi(zx)$ for all x in A^{**} and all pure states φ of A. For an abelian C*-algebra $C_0(X)$, the enveloping W*-algebra $C_0(X)^{**} = \bigoplus_\infty \{L^\infty(\mu) : \mu \in \mathcal{C}\} \oplus_\infty \ell^\infty(X)$, where $\mathcal{C}$ is a maximal family of mutually singular continuous measures on X. In this way, every x in $C_0(X)^{**}$ can be written as a direct sum $x = x_d + x_a$ of the diffuse part x_d and the atomic part x_a, and $zx = x_a \in \ell^\infty(X)$. Note that a measure μ on X is atomic if $\langle x, \mu\rangle = \int x_a d\mu = \langle zx, \mu\rangle$, or equivalently, μ is supported by z. Alternatively, atomic measures are exactly countable linear sums of point masses. In general,

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atomic positive functionals of a non-abelian C*-algebra A are countable linear sums of pure states of A ([13, 14]).

We call zA^{**} the *atomic part* of A^{**}. An element x of A^{**} is said to *have a continuous atomic part* if $zx = za$ for some a in A (cf. [18]). In this case, x and a agree on $P(A) \cup \{0\}$ since $\varphi(x) = \varphi(zx) = \varphi(za) = \varphi(a)$ for all pure states φ of A. In particular, $\varphi \longmapsto \varphi(x)$ is uniformly continuous on $P(A) \cup \{0\}$. Shultz [18] showed that x in A^{**} has a continuous atomic part whenever x, x^*x and xx^* are uniformly continuous on $P(A) \cup \{0\}$. Later, Brown [7] proved:

Theorem 1 ([7]). *Let x be an element of A^{**}. Then x has a continuous atomic part (i.e. $zx \in zA$) if and only if x is uniformly continuous on $P(A) \cup \{0\}$.*

The Stone-Weierstrass problem for C*-algebras conjectures that if B is a C*-subalgebra of a C*-algebra A separating points in $P(A) \cup \{0\}$ then $A = B$ (see *e.g.* [11]). The facial structure of the compact convex set $Q(A)$ sheds some light on solving the Stone-Weierstrass problem. The classical papers of Tomita [19, 20], Effros [12], Prosser [17], and Akemann, Andersen and Pedersen [5], among others, have been exploring the interrelationship among weak* closed faces of $Q(A)$, closed projections in A^{**} and norm closed left ideals of A, in the hope that this will help to solve the Stone-Weierstrass problem.

Recall that a projection p in A^{**} is *closed* if the face

$$F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$$

of $Q(A)$ supported by p is weak* closed (and thus weak* compact). In the abelian case $A = C_0(X)$, closed projections arise exactly from characteristic functions of closed subsets of X. Closed projections p in A^{**} are also in one-to-one correspondence with norm closed left ideals L of A via

$$L = A^{**}(1-p) \cap A.$$

Note also that the Banach double dual L^{**} of L, identified with the weak* closure of L in A^{**}, is a weak* closed left ideal of the W*-algebra A^{**}. More precisely, we have $L^{**} = A^{**}(1-p)$. Moreover, we have isometrical isomorphisms $a + L \longmapsto ap$ and $x + L^{**} \longmapsto xp$ under which

$$A/L \cong Ap \quad \text{and} \quad (A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p$$

as Banach spaces, respectively [12, 17, 1]. Similarly, we have Banach space isomorphisms between $A/(L + L')$ and pAp, and $A^{**}/(L^{**} + L^{**\prime})$ and $pA^{**}p$, respectively, where B' denotes the set $\{b^* : b \in B\}$. The significance of these objects arises from the following local versions of the Kadison function representation for pAp and Ap.

Theorem 2 ([6, 3.5],[21]).

1. *$pA_{sa}p$ (resp. $pA_{sa}^{**}p$) is isometrically order isomorphic to the Banach space of all continuous (resp. bounded) affine functions on $F(p)$ which vanish at zero.*
2. *Let xp be an element of $A^{**}p$. Then $xp \in Ap$ if and only if the affine functions $\varphi \longmapsto \varphi(x^*x)$ and $\varphi \longmapsto \varphi(a^*x)$ are continuous on $F(p)$, $\forall a \in A$. Consequently,*

$$xp \in Ap \Leftrightarrow px^*xp \in pAp \text{ and } pa^*xp \in pAp, \quad \forall a \in A.$$

Denote the extreme boundary of $F(p)$ by $X_0 = (P(A) \cup \{0\}) \cap F(p)$, which consists of all pure states of A supported by p together with the zero functional. Motivated by Theorem 1, we shall attack the following

Problem 3. *Suppose that pxp in $pA^{**}p$ is uniformly continuous on X_0, or continuous on its weak* closure, when we consider pxp as an affine functional on $F(p)$ (Theorem 2). Can we infer that pxp has a continuous atomic part as a member of $pA^{**}p$, i.e. , $zpxp = zpap$ for some a in A?*

A quite satisfactory and affirmative answer for a similar question for elements xp of the left quotient $A^{**}p$ was obtained in [10]. Utilizing the technique and repeating parts of the argument provided in [10], we will achieve positive results here as well. We will impose conditions on the closed projection p (or equivalently, geometric conditions on $F(p)$) to ensure an affirmative answer to Problem 3. We note that the counter examples in [10] indicate that our results are sharp and Problem 3 does not always have an appropriate solution in general. For the convenience of the readers, we borrow an example from [10] and present it at the end of this note.

2. The results

Let A be a C*-algebra and p a closed projection in A^{**}. Recall that A_{sa}^m consists of all limits in A_{sa}^{**} of monotone increasing nets in A_{sa} and $(A_{sa})_m = -A_{sa}^m$. While A_{sa} consists of continuous affine real-valued functions of $Q(A)$ vanishing at 0 (the Kadison function representation), the norm closure $(A_{sa}^m)^-$ of A_{sa}^m consists of *lower semicontinuous elements* and the norm closure $\overline{(A_{sa})_m}$ of $(A_{sa})_m$ consists of *upper semicontinuous elements* in A^{**}. An element x of A_{sa}^{**} is said to be *universally measurable* if for each φ in $Q(A)$ and $\varepsilon > 0$ there exist a lower semicontinuous element l and an upper semicontinuous element u in A^{**} such that $u \leq x \leq l$ and $\varphi(l - u) < \varepsilon$ [15].

We note that $pA_{sa}p$ consists of continuous affine real-valued functions on $F(p)$. It was shown in [9] that every lower (resp. upper) semicontinuous bounded affine real-valued function on $F(p)$ vanishing at 0 is the restriction of a lower (resp. upper) semicontinuous element in A_{sa}^{**} to $F(p)$; namely it is of the form pxp for some x in $(A_{sa}^m)^-$ or $\overline{(A_{sa})_m}$. Analogously, pxp in $pA_{sa}^{**}p$ is said to be *universally measurable on* $F(p)$ if for each φ in $F(p)$ and $\varepsilon > 0$, there exist an l in $(A_{sa}^m)^-$ and a u in $\overline{(A_{sa})_m}$ such that $pup \leq pxp \leq plp$ and $\varphi(l - u) < \varepsilon$. And pxp in $pA^{**}p$ is said to be universally measurable on $F(p)$ if both the real and imaginary parts of pxp are.

A Borel measure on $F(p)$ is a *boundary measure* if it is supported by the closure of the extreme boundary X_0 of $F(p)$. A boundary measure m of $F(p)$ with $\|m\| = m(F(p)) = 1$ represents a unique point ϕ in $F(p)$, where $\phi(a) = \int \psi(a) dm(\psi), \forall a \in A$. An element pxp of $pA_{sa}^{**}p$ is said to *satisfy the barycenter formula* if $\phi(x) = \int \psi(x) dm(\psi)$ whenever m is a boundary measure of $F(p)$ representing ϕ. Semicontinuous affine elements in $pA_{sa}^{**}p$ satisfy the barycenter formula, and so do universally measurable elements.

Lemma 4. *Let x be an element of A_{sa}^{**} and let $\overline{X}$ be the weak* closure of $X = F(p) \cap P(A)$ in $F(p)$. If pxp satisfies the barycenter formula and is continuous on $\overline{X}$ then $pxp \in pAp$.*

Proof. We give a sketch of the proof here, and refer the readers to [10] in which a similar result is given in full detail. In view of Theorem 2, we need only verify that $\varphi \mapsto \varphi(x)$ is weak* continuous on $F(p)$. Suppose φ_λ and φ are in $F(p)$ and $\varphi_\lambda \longrightarrow \varphi$ weak*. Since the norm of an element of $pA_{sa}p$ is determined by the pure states supported by p, we can embed $pA_{sa}p$ as a closed subspace of the Banach space $C_{\mathbb{R}}(\overline{X})$ of continuous real-valued functions defined on $\overline{X}$. Let m_λ be any positive extension of φ_λ from $pA_{sa}p$ to $C_{\mathbb{R}}(\overline{X})$ with $\|m_\lambda\| = \|\varphi_\lambda\| \leq 1$. Hence, $(m_\lambda)_\lambda$ is a bounded net in $M(\overline{X})$, the Banach dual space of $C_{\mathbb{R}}(\overline{X})$, consisting of regular finite Borel measures on the compact Hausdorff space $\overline{X}$. Then, by passing to a subnet if necessary, we have $m_\lambda \to m$ in the weak* topology of $M(\overline{X})$. Clearly, $m \geq 0$ and $m_{|pA_{sa}p} = \varphi$. Since pxp satisfies the barycenter formula and is continuous on $\overline{X}$, we have

$$\varphi_\lambda(x) = \int_{\overline{X}} \psi(x)\, dm_\lambda(\psi) = \int_{\overline{X}} \psi(pxp)\, dm_\lambda(\psi) \longrightarrow \int_{\overline{X}} \psi(pxp)\, dm(\psi) = \int_{\overline{X}} \psi(x)\, dm(\psi) = \varphi(x).$$

□

2.1. **The case where p has MSQC.** Let A be a C*-algebra. Recall that a projection p in A^{**} is closed if the face $F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$ is weak* closed. Analogously, p is said to be *compact* [2] (see also [6]) if $F(p) \cap S(A)$ is weak* closed, where $S(A) = \{\varphi \in Q(A) : \|\varphi\| = 1\}$ is the state space of A. Let p be a closed projection in A^{**}. Then h in $pA^{**}_{sa}p$ is said to be q–*continuous* [3] on p if the spectral projection $E_F(h)$ (computed in $pA^{**}p$) is closed for every closed subset F of $\mathbb{R}$. Moreover, h is said to be *strongly* q–*continuous* [6] on p if, in addition, $E_F(h)$ is compact whenever F is closed and $0 \notin F$. It is known from [6, 3.43] that h is strongly q–continuous on p if and only if $h = pa = ap$ for some a in A_{sa}. In general, h in $pA^{**}p$ is said to be *strongly* q–*continuous* on p if both Reh and Imh are.

Denote by $SQC(p)$ the C*-algebra of all strongly q–continuous elements on p. We say that p has MSQC ("*many strongly* q–*continuous elements*") if $SQC(p)$ is σ–weakly dense in $pA^{**}p$. Brown [8] showed that p has MSQC if and only if $pAp = SQC(p)$ if and only if pAp is an algebra. In particular, every central projection p (especially, $p = 1$) has MSQC. We provide a partial answer to Problem 3 by the following:

Theorem 5. *Let p have* MSQC *and x be in A^{**}. Let $X_0 = (F(p) \cap P(A)) \cup \{0\}$ be the extreme boundary of $F(p)$. Then $zpxp \in zpAp$ if and only if pxp is uniformly continuous on X_0.*

Proof. The necessities are obvious and we check the sufficiency. Note that pAp is now a C*-algebra with the pure state space $P(pAp) = F(p) \cap P(A)$. The maximal atomic projection of pAp is zp. By Theorem 1, $zpxp$ belongs to $zpAp$ whenever it is uniformly continuous on X_0. □

Corollary 6. *Let p have* MSQC *and x be in A^{**}. If pxp is continuous on $\overline{X} = \overline{F(p) \cap P(A)}$ then $zpxp \in zpAp$.*

Proof. We simply note that either 0 belongs to $\overline{X}$ or 0 is isolated from $X = F(p) \cap P(A)$ in $X_0 = (F(p) \cap P(A)) \cup \{0\}$. Consequently, continuity on the compact set $\overline{X}$ ensures uniform continuity on X_0. □

2.2. **The case where p is semiatomic .** Let A be a C*-algebra and p a closed projection in A^{**}. Recall that A is said to be scattered [13, 14] if $Q(A) \subseteq zQ(A)$ and p is said to be atomic [8] if $F(p) \subseteq zF(p)$. If A is scattered then every closed projection in A^{**} is atomic. Moreover, A is said to be semiscattered [4] if $\overline{P(A)} \subseteq zQ(A)$. Analogously, we say that a closed projection p is *semiatomic* if the weak* closure of $F(p) \cap P(A)$ contains only atomic positive linear functionals of A, *i.e.* $\overline{F(p) \cap P(A)} \subseteq zF(p)$. It is easy to see that if A is semiscattered then every closed projection in A^{**} is semiatomic.

The following is a generalization of [7, Theorem 6] in which $p = 1$.

Lemma 7 ([10])**.** *Let x in $zpA^{**}p$ be uniformly continuous on $X_0 = (F(p) \cap P(A)) \cup \{0\}$. Then x is in the C*-algebra B generated by $zpAp$. In particular, $x = zy$ for some universally measurable element y of $pA^{**}p$.*

We provide another partial answer to Problem 3 by the following

Theorem 8. *Let p be semiatomic and x be in A^{**}. Let $\overline{X} = \overline{F(p) \cap P(A)}$. Then $zpxp \in zpAp$ if and only if pxp is continuous on $\overline{X}$.*

Proof. We prove the sufficiency only. Let x in A^{**} satisfy the stated condition. Since $zpxp$ is uniformly continuous on $X_0 = (P(A) \cap F(p)) \cup \{0\}$, by Lemma 7, there is a universally measurable element y of $pA^{**}p$ such that $zpxp = zy$. Since p is assumed to be semiatomic, each φ in $\overline{X} = \overline{P(A) \cap F(p)}$ is atomic and thus $\varphi(x) = \varphi(zpxp) = \varphi(zy) = \varphi(y)$. In particular, the universally measurable element y is continuous on $\overline{X}$. It follows from Lemma 4 that $y \in pAp$. As a consequence, $zpxp \in zpAp$. □

Example 9 (The full version appeared in [10])**.** This example tells us that p having MSQC is necessary in Theorem 5 and continuity on $\overline{X}$ is necessary in Theorem 8.

Let A be the scattered C*-algebra of sequences of 2×2 matrices $x = (x_n)_{n=1}^{\infty}$ such that

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \longrightarrow x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ entrywise,}$$

and equipped with the ℓ^∞–norm. Note that the maximal atomic projection $z = 1$ in this case. Let

$$p_n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, n = 1, 2, \ldots, \quad \text{and} \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $p = (p_n)_{n=1}^{\infty}$ is a closed projection in A^{**}. We claim that p does *not* have MSQC. In fact, suppose $x = (x_n)_{n=1}^{\infty}$ in A is given by

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

such that $x_n \to x_\infty$. Then $(pxp)_n = \lambda_n p_n$, $n = 1, 2, \ldots$, and $(pxp)_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $\lambda_n = \frac{a_n + b_n + c_n + d_n}{2} \to \frac{a+d}{2}$. Consequently, $(pxp)_n^2 = \lambda_n^2 p_n$, $n = 1, 2, \ldots$, and $(pxp)_\infty^2 = \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}$. If

$(pxp)^2 \in pAp$, we must have $\lambda_n^2 \to \frac{a^2+d^2}{2}$. This occurs exactly when $a = d$. In particular, pAp is not an algebra and thus p does *not* have MSQC.

On the other hand, the set $X = P(A) \cap F(p)$ of all pure states in $F(p)$ consists exactly of φ_n, ψ_1 and ψ_2 which are given by

$$\varphi_n(x) = \mathrm{tr}(x_n p_n), \quad n = 1, 2, \ldots,$$

and

$$\psi_1(x) = a, \quad \psi_2(x) = d,$$

where $x = (x_n)_{n=1}^{\infty} \in A$ and $x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Since $\varphi_n \to \frac{1}{2}(\psi_1 + \psi_2) \neq 0$, $X_0 = X \cup \{0\}$ is discrete. Consider $y = (y_n)_{n=1}^{\infty}$ in A^{**} given by

$$y_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ n = 1, 2, \ldots, \quad \text{and} \quad y_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, the universally measurable element pyp is uniformly continuous on X_0 but $pyp \notin pAp$. □

References

[1] C. A. Akemann, *Left ideal structure of C*-algebras*, J. Funct. Anal. **6** (1970), 305–317.
[2] ______, *A Gelfand representation theory for C*-algebras*, Pac. J. Math. **39** (1971), 1-11.
[3] ______, G. K. Pedersen and J. Tomiyama, *Multipliers of C*-algebras*, J. Funct. Anal. **13** (1973), 277-301.
[4] ______ and F. Shultz, *Perfect C*-algebras*, Memoirs A. M. S. **326**, 1985.
[5] ______, J. Andersen and G. K. Pedersen, *Approaching to infinity in C*-algebras*, J. Operator Theory **21** (1989), 252–271.
[6] L. G. Brown, *Semicontinuity and multipliers of C*-algebras*, Can. J. Math. XL (1988), no. 4, 865–988.
[7] ______, *Complements to various Stone-Weierstrass theorems for C*-algebras and a theorem of Shultz*, Commun. Math. Phys. **143** (1992), 405–413.
[8] ______, *MASA's and certain type I closed faces of C*-algebras*, preprint.
[9] ______, *Semicontinuity and closed faces of C*-algebras*, unpublished notes.
[10] ______ and Ngai-Ching Wong, *Left quotients of a C*-algebra, II: Atomic parts of left quotients*, J. Operator Theory **44** (2000), 207–222.
[11] J. Diximier, *C*-algebras*, North-Holland publishing company, Amsterdam–New York–Oxford, 1977.
[12] E. G. Effros, *Order ideals in C*-algebras and its dual*, Duke Math. **30** (1963), 391–412.
[13] H. E. Jensen, *Scattered C*-algebras*, Math. Scand. **41** (1977), 308–314.
[14] ______, *Scattered C*-algebras, II*, Math. Scand. **43** (1978), 308–310.
[15] G. K. Pedersen, *Applications of weak* semicontinuity in C*-algebra theory*, Duke Math. J. **39** (1972), 431–450.
[16] ______, *C*-algebras and their automorphism groups*, Academic Press, London, 1979.
[17] R. T. Prosser, *On the ideal structure of operator algebras*, Memoirs A. M. S. **45**, 1963.
[18] F. W. Shultz, *Pure states as a dual object for C*-algebras*, Commun. Math. Phys. **82** (1982), 497–509.
[19] M. Tomita, "Spectral theory of operator algebras, I", *Math. J. Okayama Univ.* **9** (1959), 63–98.
[20] ______, "Spectral theory of operator algebras, II", *Math. J. Okayama Univ.* **10** (1960), 19–60.
[21] Ngai-Ching Wong, *Left quotients of a C*-algebra, I: Representation via vector sections*, J. Operator Theory **32**, 1994, 185–201.

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