SPACE-FILLING CURVES AND HAUSDORFF DIMENSIONS

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Abstract. We construct a continuous curve from the interval $[0,1]$ into the n-dimensional cube $[0,1]^n$ in $\mathbb{R}^n$ under which the entire cube is filled by the image of a subset of $[0,1]$ of Hausdorff dimension r, for any positive integer $n \geq 2$ and any real number r between 0 and 1.

A *space-filling curve* is a continuous function from the one-dimensional unit interval $[0,1]$ onto the two dimensional unit square $[0,1]^2 = [0,1] \times [0,1]$. In 1879, Netto showed that $[0,1]^2$ cannot be a homeomorphic image of $[0,1]$. Hence the discovery of a space-filling curve (and consequently the fact that $[0,1]^2$ is a continuous image of $[0,1]$) is surprising. After Peano's first example given in 1890, Hilbert in 1891, Lebesgue in 1904, Schoenberg in 1938, and many others, subsequently simplified and formalized the construction of space-filling curves. As a uniform limit of a sequence of simple polygons Peano's curve is very intuitive and visible, although it is difficult to compute. On the other hand, whereas Schoenberg's curve is given by a fixed formula, the geometrical content of that formula is not easy to observe. The visibility and computability of those curves of Hilbert and Lebesgue are somewhere in between. See, for example, Sagan [4] for a brief history of this subject.

In 1927, Hausdorff and Alexandroff showed that every compact set is a continuous image of a dyadic discontinuum (see e.g. [5, p. 116]). In 1928, Hahn simplified his earlier argument to prove that every connected and locally connected compact set is a continuous image of the unit interval (see e.g. [5]). Consequently, all n–dimensional unit cubes $[0,1]^n$ in $\mathbb{R}^n$ are continuous images of the unit interval $[0,1]$. On the other hand, we note that the space-filling curve of Lebesgue maps the Cantor set onto $[0,1]^2$. In other words, $[0,1]^2$ is a continuous image of a subset of $[0,1]$ of Hausdorff dimension $\log 2/\log 3$ in this case. Recently, Liu [3] constructed explicitly a space-filling curve under which $[0,1]^2$ is a continuous image of a zero dimensional subset of $[0,1]$. By a similar architecture, we shall construct in this note computable and visible curves in $\mathbb{R}^n$ under which the n-dimensional unit cube $[0,1]^n$ is a continuous image of a subset of the unit interval $[0,1]$ of Hausdorff dimension r for any positive integer $n \geq 2$ and any real number r between 0 and 1. Our curves emphasis on both geometrical clarity and computing effectiveness, as one can see from Example 3 in which a space-filling curve mapping a subset of $[0,1]$ of Hausdorff dimension $\log 4/\log 6$ onto $[0,1]^2$ is constructed.

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We begin with an elementary lemma which says that each real number $G \geq 1$ is the geometric mean of a sequence of positive integers. In the following, we denote by $[a]$ the greatest integral part of a real number a.

Lemma 1. *For each real number $G \geq 1$, there exists a sequence of positive integers $\{q_i\}$ such that*

$$\lim_{k\to\infty} (q_1 \cdots q_{2^k})^{\frac{1}{2^k}} = G,$$

where $q_i \in \{[G], [G]+1\}$ for $i = 1, 2, \ldots, 2^k$.

Proof. Suppose a real number $a = \overline{a}.a_1a_2a_3 \cdots \geq 1$ in base 2 expansion is given, where $\overline{a} = [a]$. In other words, $a = \overline{a} + \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \cdots$. Define for each positive integer k a function $GM_k : \mathbb{R} \to \mathbb{R}$ by $GM_k(a) = (q_1 \cdots q_{2^k})^{\frac{1}{2^k}}$ where q_i's are chosen in the following way. Let $d_0 = 0$. For each positive integer i, let $d_i = 2^{i-1}a_1 + 2^{i-2}a_2 + \cdots + a_i$ and set $q_{2^{i-1}+1} = \cdots = q_{2^{i-1}+d_i-d_{i-1}} = \overline{a}+1$ and $q_{2^{i-1}+d_i-d_{i-1}+1} = \cdots = q_{2^i} = \overline{a}$. Observe that

$$\begin{aligned} GM_k(a) &= (q_1 \cdots q_{2^k})^{\frac{1}{2^k}} = (\overline{a})^{\frac{2^k - d_k}{2^k}} (\overline{a}+1)^{\frac{d_k}{2^k}} \\ &= \overline{a}(\frac{\overline{a}+1}{\overline{a}})^{\frac{d_k}{2^k}} = \overline{a}(\frac{\overline{a}+1}{\overline{a}})^{\frac{a_1}{2}+\cdots+\frac{a_k}{2^k}}. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} GM_k(a) = \overline{a}(\frac{\overline{a}+1}{\overline{a}})^{a-\overline{a}}.$$

Now for any real number $G \geq 1$, we take $I = [G]$ and $r = \frac{\log \frac{G}{I}}{\log \frac{I+1}{I}}$. Then $0 \leq r < 1$ and

$$I(\frac{I+1}{I})^r = G.$$

By putting $a = I + r$, we can get positive integers q_i's from $[G]$ and $[G]+1$ such that

$$\lim_{k\to\infty} (q_1 \cdots q_{2^k})^{\frac{1}{2^k}} = \overline{a}(\frac{\overline{a}+1}{\overline{a}})^{a-\overline{a}} = G.$$

□

Theorem 2. *Let $n \geq 2$ be any positive integer and r any real number such that $0 < r < 1$. There exists a continuous curve from $[0,1]$ into $[0,1]^n$ under which the entire n-dimensional unit cube $[0,1]^n$ is the image of a subset A of $[0,1]$ of Hausdorff dimension r.*

Proof. First, we assume $r \leq \log 2^n / \log(2^n + 2)$ and fix a real number $G \geq 2^n + 2$ such that

$$r = \frac{\log 2^n}{\log G}.$$

By Lemma 1, we have a sequence of integers $\{q_i\}$ such that $q_i \geq 2^n + 2$ for $i = 1, 2, 3, \ldots$ and $\lim_{k\to\infty}(q_1 \cdots q_{2^k})^{\frac{1}{2^k}} = G$. Let $A, A_1, A_2, A_3, \ldots$ be subsets of the interval $[0,1]$ such that for

$l = 1, 2, 3, \ldots,$

$$A_l = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{q_1 \cdots q_k} : t_k = 1, 2, 3, \ldots, 2^n,\ k = 1, 2, 3, \ldots, 2^l \right\}$$

and

$$A = \cap_{l=1}^{\infty} A_l = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{q_1 \cdots q_k} : t_k = 1, 2, 3, \ldots, 2^n,\ k = 1, 2, 3, \ldots \right\}.$$

Since A_l is a disjoint union of $(2^n)^{2^l}$ intervals, the Hausdorff p-dimensional measure of A_l for any $p > 0$ is

$$H_p^*(A_l) = (2^n)^{2^l} \left(\frac{1}{q_1 \cdots q_{2^l}} \right)^p, \quad l = 1, 2, \ldots.$$

Thus

$$H_p^*(A) = \lim_{l \to \infty} H_p^*(A_l) = \lim_{l \to \infty} \left[\frac{2^n}{(q_1 \cdots q_{2^l})^{p/2^l}} \right]^{2^l}.$$

Consequently, the Hausdorff dimension of A is

$$D_H(A) = \inf\{p > 0 : H_p^*(A) = 0\} = \frac{\log 2^n}{\log G} = r.$$

The desired curve is given by sending t in $[0, 1]$ to the point $(x(t), y(t), \ldots, z(t))$ in $[0, 1]^n$, where

$$t = \sum_{k=1}^{\infty} \frac{t_k}{q_1 q_2 \cdots q_k}, \quad t_k \in \{0, 1, 2, \ldots, q_k - 1\}, \tag{1}$$

$$\left. \begin{array}{l} x(t) = 0.x_1 x_2 \cdots \\ y(t) = 0.y_1 y_2 \cdots \\ \vdots \\ z(t) = 0.z_1 z_2 \cdots \end{array} \right\} \text{in base 2 expansion}$$

(in other words, $x(t) = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots$, $y(t) = \frac{y_1}{2} + \frac{y_2}{2^2} + \cdots$, $\ldots$, $z(t) = \frac{z_1}{2} + \frac{z_2}{2^2} + \cdots$ where $x_i, y_i, \ldots, z_i \in \{0, 1\}$ and $t_0 = x_0 = y_0 = \cdots = z_0 = 0$) such that for $k \geq 1$,

$$(x_k, y_k, \ldots, z_k)$$

$$= \begin{cases} (t_k^1, t_k^2, \ldots, t_k^n), & \text{if } 1 \leq t_k \leq 2^n \\ & \quad \text{and } t_k - 1 = t_k^1 t_k^2 \cdots t_k^n \text{ in base 2 expansion;} \\ (1, 1, \ldots, 1), & \text{if } 2^n + 1 \leq t_k \leq q_k - 2; \\ (x_{k-1}, y_{k-1}, \ldots, z_{k-1}), & \text{if } t_k = 0 = t_{k-1} \text{ or } q_k - t_k = 1 = q_{k-1} - t_{k-1}; \\ (1 - x_{k-1}, 1 - y_{k-1}, \ldots, 1 - z_{k-1}), & \text{if } t_k = 0 \neq t_{k-1} \text{ or } q_k - t_k = 1 \neq q_{k-1} - t_{k-1}. \end{cases}$$

A routine verification will show that even for those t having two distinct expansions in (1) the values of $x(t)$, $y(t)$, $\ldots$, $z(t)$ are unique. The continuity of $x(t)$, $y(t)$, $\ldots$, $z(t)$ follows easily from the fact that x_k, y_k, $\ldots$, z_k depend on the first k digits of t for all $k = 1, 2, \ldots$. It is also plain that the image of A under this curve is the entire square $[0, 1]^n$.

Finally, if $r > \log 2^n/\log(2^n+2)$ we may choose a bigger dimension m such that $r \leq \log 2^m/\log(2^m+2)$ and project the curve thus obtained from $[0,1]^m$ onto $[0,1]^n$ in a trivial manner. □

It is not difficult to see that a similar result for the case $r = 0$ can be obtained by taking $q_k \longrightarrow \infty$ in the above proof. In fact, Liu got his space-filling curve in this way in [3]. On the other hand, we refer the readers to [5] for the classical case $r = 1$. Moreover, a key step in the above proof is to establish a Cantor-like subset A of $[0,1]$ with a given Hausdorff dimension r. In general, we learn from an exercise in [1, p. 157] that there exists a metric space S with Hausdorff dimension r for any positive real number r, and another one in [2, p. 35] that there is a totally disconnected subset of the plane of Hausdorff dimension r for every real number r between 0 and 2.

Example 3. The simplest example of our construction is a space-filling curve under which $[0,1]^2$ is a continuous image of a subset A of $[0,1]$ of Hausdorff dimension $\log 4/\log 6$. This can be obtained by setting $q_1 = q_2 = \cdots = 6$ and $n = 2$ in the proof of Theorem 2. More precisely, the space-filling curve $t \mapsto (x(t), y(t))$ is given by writing

$$t = 0.t_1t_2\cdots \quad \text{in base 6 expansion}$$

and

$$\left.\begin{array}{l} x(t) = 0.x_1x_2\cdots \\ y(t) = 0.y_1y_2\cdots \end{array}\right\} \text{ in base 2 expansion}$$

(in particular, $t_0 = x_0 = y_0 = 0$) such that for $k \geq 1$,

$$(x_k, y_k) = \begin{cases} (0,0), & \text{if } t_k = 1; \\ (0,1), & \text{if } t_k = 2; \\ (1,0), & \text{if } t_k = 3; \\ (1,1), & \text{if } t_k = 4; \\ (x_{k-1}, y_{k-1}), & \text{if } t_k = 0 = t_{k-1} \text{ or } t_k = 5 = t_{k-1}; \\ (1-x_{k-1}, 1-y_{k-1}), & \text{if } t_k = 0 \neq t_{k-1} \text{ or } t_k = 5 \neq t_{k-1}. \end{cases}$$

In general, the first k digits (in base 2) of $x(t)$ and $y(t)$ can be calculated in terms of the first k digits (in base 6) of t. The image of

$$A = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{6^k} : t_k = 1,2,3,4,\ k = 1,2,\ldots \right\}$$

fills up the entire square $[0,1]^2$.

In Figure 1, we draw four polygons each of which approximates this space-filling curve within 1/2, 1/4, 1/8 and 1/16 uniformly in both x- and y-directions, respectively. They are obtained by making linear interpolation for the sets of data consisting of first one, two, three and four digits of t, $x(t)$ and $y(t)$, respectively (in which we represent $1 = 0.55\cdots$ in base 6 for convenience). For example, the approximating polygon of order 2 at the upper right corner of

Figure 1 is plotted based on the 36 data $(t, x(t), y(t))$ from the set

$$\{(0.00, 0.00, 0.00), (0.01, 0.00, 0.00), (0.02, 0.00, 0.01), (0.03, 0.01, 0.00), \ldots, (0.53, 0.11, 0.10), (0.54, 0.11, 0.11), (0.55, 0.11, 0.11)\}.$$

While the digital definition of the curve is effective in calculation, it also introduces some similarities among the approximating polygons. For example, the approximating polygon of order 3 at the lower left corner of Figure 1, which is plotted by using first 3 digits of t, $x(t)$ and $y(t)$, can be obtained easily by first shrinking the approximating polygon of order 2 to one half of its size, then copying the result four times in a square pattern, and finally connecting the four parts suitably by joins. A similar procedure gives the approximating polygon of order 4 (via that of order 3) at the lower right corner of Figure 1.

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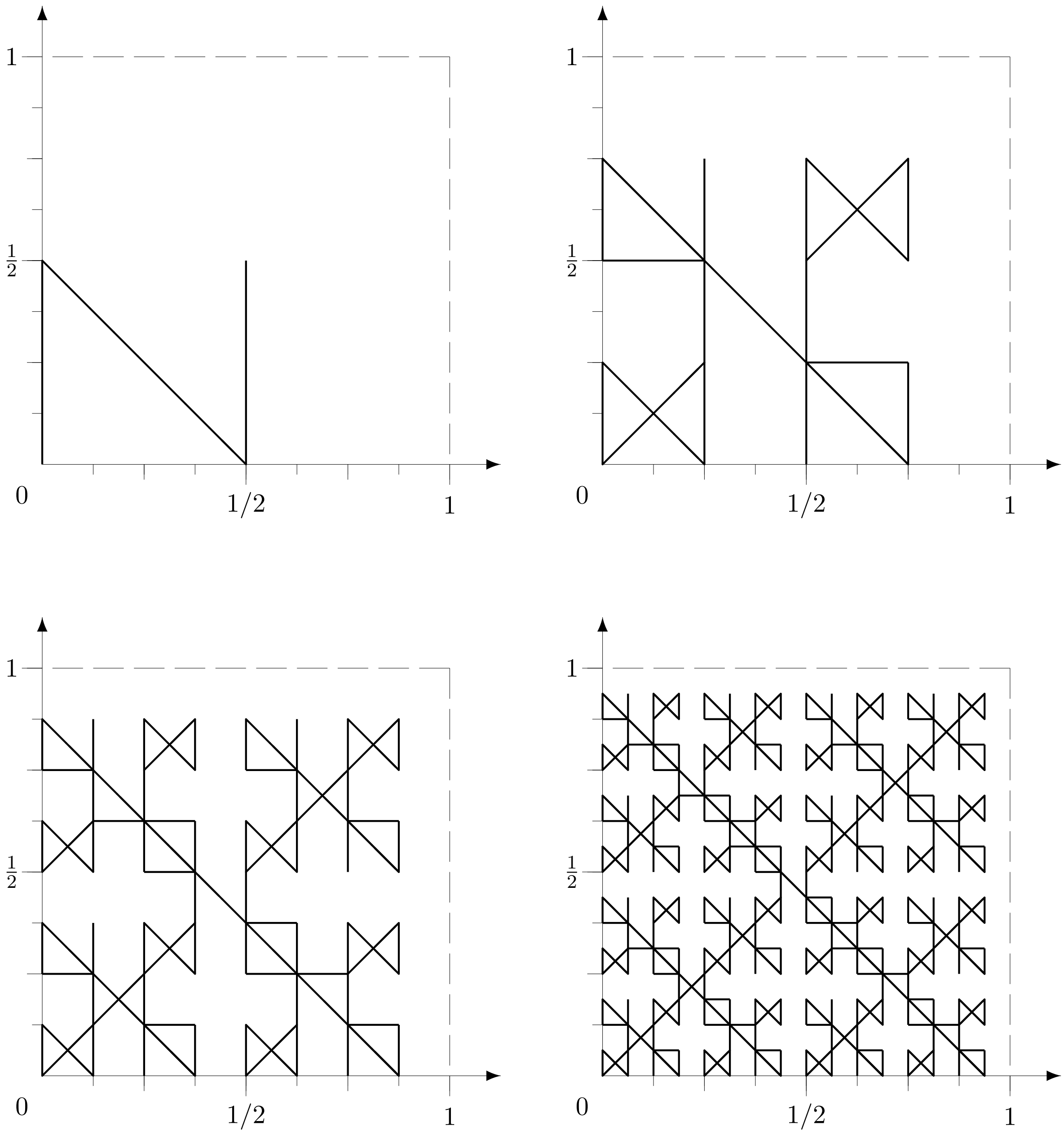

FIGURE 1. Approximating polygons of order 1 (upper left), 2 (upper right), 3 (lower left) and 4 (lower right) of a space-filling curve.