CONSTRUCTING SPACE-FILLING CURVES OF COMPACT CONNECTED MANIFOLDS

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Abstract. Let M be a compact connected (topological) manifold of finite or infinite dimension n. Let $0 \leq r \leq 1$ be arbitrary but fixed. We construct in this paper a space-filling curve f from $[0,1]$ onto M, under which M is the image of a compact set A of Hausdorff dimension r. Moreover, the restriction of f to A is one-to-one over the image of a dense subset provided that $0 \leq r \leq \frac{\log 2^n}{\log(2^n+2)}$. The proof is based on the special case where M is the Hilbert cube $[0,1]^\omega$.

1. Introduction

Following the first example given by Peano in 1890, we know that every n-dimensional cube $[0,1]^n$ has a space-filling curve (see, e.g. [11]). In other words, $[0,1]^n$ is a continuous image of the unit interval $[0,1]$. This fact is eventually generalized to give

Theorem 1 (Mazurkiewicz and Hahn, (see e.g. [11, p. 106])). *Let X be a metrizable space. Then X is a continuous image of $[0,1]$ if and only if X is compact, connected, and locally connected.*

As a consequence of Theorem 1, in addition to finite dimensional cubes $[0,1]^n, n = 1,2,\ldots$, the Hilbert cube $\mathbb{H} = [0,1]^\omega$, i.e. the product space of countably infinitely many copies of $[0,1]$, also has a space-filling curve. It is known that every separable infinite dimensional compact convex set in a Fréchet space is affinely homeomorphic to $\mathbb{H}$ (see, e.g. [1, p. 100] or [8, p. 40]). Consequently, there are also space-filling curves of such spaces.

A metric space M is called a *Hilbert cube manifold* if for each x in M, there is a base of neighborhoods of x in which every member is homeomorphic to an open subset of $\mathbb{H}$ (see, e.g. [1, p. 298]). When M is compact, it is equivalent to saying that there exist compact subsets $U_1, \ldots, U_k$ of M such that M is covered by the interiors of $U_1, \ldots, U_k$ and each of them is homeomorphic to $\mathbb{H}$. In this paper, compact (topological) manifolds M are either modeled on $[0,1]^n$ if $\dim M = n < \infty$, or modeled on $\mathbb{H} = [0,1]^\omega$ if $\dim M = \infty$.

The existence of a space-filling curve of any compact connected manifold is ensured by Theorem 1. In this paper, we shall construct a *computable* space-filling curve f of the Hilbert cube $\mathbb{H}$. Similar results has been obtained for finite dimensional cubes $[0,1]^n$ in [5] for $n = 1,2,\ldots$. In [3], M. Bestvina and J. J. Walsh showed that for positive integers $n > m \geq 2$, there is a surjective continuous function g from $\mathbb{R}^n$ onto $\mathbb{R}^m$ that is one-to-one

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over the image of a dense subset. Moreover, if $\mathbb{R}^m = g(K)$ for any σ-compact subset K of $\mathbb{R}^n$ then $\dim K = n$. In our construction, for any pre-assigned r between 0 and 1, we can construct explicitly a space-filling curve f from $[0,1]$ onto $[0,1]^n$, $n = 1, 2, \ldots, \omega$, maps a compact set A of dimension r onto $[0,1]^n$. Moreover, the restriction of f to A is one-to-one over the image of a dense subset provided $0 \leq r \leq \frac{\log 2^n}{\log(2^n+2)}$. Similar conclusions are carried to compact connected manifolds, which supplement the results in [2, 3, 4].

There is a variety of applications of space-filling curves. To name a few, we mention [6] for embedding Urysohn space into $C[0,1]$, [12] for classifying geometric finiteness of Kleinian groups, and [9] for converting integral equations in n variables into one involving one variable. See also [13] for more interesting information.

2. Main results

Recall that the Hilbert cube $\mathbb{H}$ can be embedded into the separable Hilbert space l_2 as the set $\{(x_n) : 0 \leq x_n \leq \frac{1}{n}\}$ in norm topology (see, e.g. [1, p. 100]). For computational ease, we identify $\mathbb{H}$ as the norm compact convex set $\{(x_n) : 0 \leq x_n \leq \frac{1}{2^{n-1}}\}$ in l_2, and frequently write $\mathbb{H} = \prod_{n=1}^{\infty} [0, \frac{1}{2^{n-1}}]$ in l_2 if no confusion arises.

Lemma 2. *We can construct a space-filling curve f of the Hilbert cube $\mathbb{H}$, under which $\mathbb{H}$ is the image of a compact subset A of $[0,1]$ of Hausdorff dimension zero. Moreover, the restriction of f to A is one-to-one over the image of a dense subset.*

Proof. We take a sequence of integers $\{q_k\}$ such that $q_k \geq 2^k + 2$ for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} \frac{k}{\log_2 q_k} = 0$. Let $A_1, A_2, A_3 \ldots$ be compact subsets of the interval $[0,1]$ defined by

$$A_l = \{\sum_{k=1}^{\infty} \frac{t_k}{q_1 \cdots q_{2^k}} : \ t_k = 1, 2, 3, \ldots, 2^k, \quad k = 1, 2, 3, \ldots, 2^l\}$$

for all $l = 1, 2, 3, \ldots$. Observe that

$$A = \bigcap_{l=1}^{\infty} A_l = \{\sum_{k=1}^{\infty} \frac{t_k}{q_1 \cdots q_{2^k}} : \ t_k = 1, 2, 3, \ldots, 2^k, \quad k = 1, 2, 3, \ldots\}$$

is compact. Since A_l is a disjoint union of $2 \times 2^2 \times 2^3 \times \cdots \times 2^{2^l} = 2^{(2^l+1)2^{l-1}}$ intervals each of length $\frac{1}{q_1 \cdots q_{2^l}}$, the Hausdorff p-dimensional measure of A_l for any $p > 0$ is

$$H_p^*(A_l) = 2^{(2^l+1)2^{l-1}} (\frac{1}{q_1 \cdots q_{2^l}})^p, \quad l = 1, 2, 3, \ldots.$$

Thus,

$$H_p^*(A) = \lim_{l\to\infty} H_p^*(A_l) = \lim_{l\to\infty} \frac{2 \cdot 2^2 \cdots 2^{2^l}}{q_1^p \cdot q_2^p \cdots q_{2^l}^p}.$$

Let $\epsilon(k) = \frac{k}{\log_2 q_k}$. Then $k = \log_2 q_k^{\epsilon(k)}$, or $2^k = q_k^{\epsilon(k)}$. Since $\epsilon(k) \to 0^+$ and $q_k \to \infty$ as $k \to \infty$, we have

$$\frac{2^k}{q_k^p} = \frac{q_k^{\epsilon(k)}}{q_k^p} = q_k^{\epsilon(k)-p} \to 0 \ \text{ if } \ p > 0.$$

Consequently, the Hausdorff dimension of A is

$$\dim A = \inf\{p > 0 : H_p^*(A) = 0\} = 0.$$

Our desired space-filling curve $f : [0,1] \to \mathbb{H}$ is given by sending t in $[0,1]$ to the point $(x_1(t), x_2(t), x_3(t), \dots)$ in $\mathbb{H} = \prod_{n=1}^{\infty} [0, \frac{1}{2^{n-1}}] \subseteq l_2$. More precisely, we write t in its *q-expansion* $t = \sum_{k=1}^{\infty} \frac{t_k}{q_1 \cdots q_k}$ where t_k belongs to $\{0, 1, 2, \dots, q_k - 1\}$, and write

$$\left.\begin{array}{rllll} x_1(t) = & 0.x_{11} & x_{12} & x_{13} \cdots \\ x_2(t) = & 0.0 & x_{22} & x_{23} \cdots \\ x_3(t) = & 0.0 & 0 & x_{33} \cdots \\ \vdots & & & \end{array}\right\} \text{ in base 2 expansion.}$$

Denote by $(a)_2$ the base 2 representation of a. We assign $q_0 = t_0 = x_{nk} = 0$ for $k = 0, 1, 2, \dots, n-1$, where $n = 1, 2, \dots$, and

$$x_{11} = \begin{cases} y_1 & \text{if } 1 \le t_1 \le 2^1,\ (t_1 - 1)_2 = y_1, \\ 1 & \text{if } 2^1 + 1 \le t_1 \le q_1 - 2, \\ 0 & \text{if } t_1 = 0 = t_0 \text{ or } q_1 - t_1 = 1 = q_0 - t_0, \\ 1 & \text{if } t_1 = 0 \ne t_0 \text{ or } q_1 - t_1 = 1 \ne q_0 - t_0; \end{cases}$$

$$(x_{12}, x_{22}) = \begin{cases} (y_1, y_2) & \text{if } 1 \le t_2 \le 2^2,\ (t_2 - 1)_2 = y_1 y_2, \\ (1,1) & \text{if } 2^2 + 1 \le t_2 \le q_2 - 2, \\ (x_{11}, 0) & \text{if } t_2 = 0 = t_1 \text{ or } q_2 - t_2 = 1 = q_1 - t_1, \\ (1 - x_{11}, 1) & \text{if } t_2 = 0 \ne t_1 \text{ or } q_2 - t_2 = 1 \ne q_1 - t_1; \end{cases}$$

$$\vdots$$

In general,

$$(x_{1n}, x_{2n}, \dots, x_{nn})$$

$$= \begin{cases} (y_1, y_2, \dots, y_n) & \text{if } 1 \le t_n \le 2^n, \text{ and} \\ & \quad (t_n - 1)_2 = y_1 y_2 \cdots y_n, \\ (1, 1, \dots, 1) & \text{if } 2^n + 1 \le t_n \le q_n - 2, \\ (x_{1\,n-1}, x_{2\,n-1}, \dots, x_{n-1\,n-1}, 0) & \text{if } t_n = 0 = t_{n-1} \text{ or} \\ & \quad q_n - t_n = 1 = q_{n-1} - t_{n-1}, \\ (1 - x_{1\,n-1}, 1 - x_{2\,n-1}, \dots, 1 - x_{n-1\,n-1}, 1) & \text{if } t_n = 0 \ne t_{n-1} \text{ or} \\ & \quad q_n - t_n = 1 \ne q_{n-1} - t_{n-1}. \end{cases}$$

A routine verification will show that even for those t having two distinct q-expansions, the values of $x_1(t)$, $x_2(t)$, $x_3(t)$, $\dots$ are unique. We check that f is (uniformly) continuous on $[0,1]$. For $\epsilon > 0$, fix a positive integer n such that

$$\sum_{k=n+1}^{\infty} (\frac{1}{2^{k-1}})^2 < \frac{\epsilon}{2}.$$

For x in $\mathbb{H}$, write $x = (x_1, x_2, \ldots, x_n, \ldots)$ in l_2. Observe that

$$\|x\|_2^2 = \sum_{k=1}^{\infty} x_k^2 < \sum_{k=1}^{n} x_k^2 + \frac{\epsilon}{2}.$$

Let m be a positive integer such that $\frac{n}{2^m} < \frac{\epsilon}{2}$. Let $\delta = \frac{1}{q_1 q_2 \cdots q_{m+1}}$. Suppose $t,\ t' \in [0,1]$ such that $|t - t'| < \delta$. We write $t,\ t'$ in their q-expansions with infinitely many nonzero digits t_k and t'_k. In this way, $t_k = t'_k$ for $k = 1, 2, \ldots, m$. Let $x = f(t)$ and $x' = f(t')$. The first m digits of x_k and x'_k agree, and thus $|x_k - x'_k| \leq \frac{1}{2^m}$, for $k = 1, 2, \ldots$. Then

$$\|x - x'\|_2^2 < \sum_{k=1}^{n} |x_k - x'_k|^2 + \frac{\epsilon}{2} \leq \frac{n}{2^m} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It is plain that the image of A under this curve is the entire of $\mathbb{H}$.

Finally, let $\mathbb{H}_0$ be the subset of $\mathbb{H}$ consisting of points x such that $f^{-1}(x)$ contains more than one points in A. Let $A_0 = f^{-1}(\mathbb{H}_0) \cap A$. It is not difficult to see that a point $x = (x_1, x_2, \ldots) \in \mathbb{H}_0$ if and only if at least one coordinate x_i has a finite binary expansion. Correspondingly, the q-expansion of any point t in A_0, when $f(t) = x$, will have a special form $t = \sum_{k=1}^{\infty} \frac{t_k}{q_1 q_2 \cdots q_k}$ in which the ith digits of the binary expansion of $t_k - 1$ are eventually constant as $k \to \infty$. Obviously, $A \backslash A_0$ is dense in A, $\mathbb{H} \backslash \mathbb{H}_0$ is dense in $\mathbb{H}$, and f is one-to-one from $A \backslash A_0$ onto $\mathbb{H} \backslash \mathbb{H}_0$. □

In the following, we denote by $[a]$ the greatest integer part of a real number a.

Lemma 3. *For each real number $G \geq 1$, there exists a sequence of positive integers $\{q_k\}$, chosen from $\{[G], [G]+1\}$, such that*

$$\lim_{k \to \infty} (q_1 q_2 \cdots q_k)^{\frac{1}{k}} = G.$$

Proof. Set $q_1 = [G]$. We shall choose subsequent q_k to satisfy the inequalities

$$[G]\ G^{k-1} \leq q_1 q_2 \cdots q_k \leq ([G]+1) G^{k-1}.$$

Suppose $q_1, q_2, \ldots, q_{k-1}$ are chosen accordingly. In case $q_1 q_2 \cdots q_{k-1} \geq G^{k-1}$, we set $q_k = [G]$; otherwise, we set $q_k = [G] + 1$. It is easy to see that q_k does not violate the above inequalities. Finally, we observe that

$$(\frac{[G]}{G})^{\frac{1}{k}} \leq \frac{(q_1 q_2 \cdots q_k)^{\frac{1}{k}}}{G} \leq (\frac{[G]+1}{G})^{\frac{1}{k}}$$

for all $k = 1, 2, \ldots$. Hence, $\lim_{k \to \infty} (q_1 q_2 \cdots q_k)^{\frac{1}{k}} = G$. □

Lemma 4. *For $0 < r \leq 1$, we can construct a space-filling curve f of the Hilbert cube $\mathbb{H}$, under which $\mathbb{H}$ is the image of a compact subset A of $[0,1]$ of Hausdorff dimension r. Moreover, the restriction of f to A is one-to-one over the image of a dense subset.*

Proof. Let $G = 2^{1/r} \geq 2$. Utilizing Lemma 3, we get a sequence $\{p_k\}$ of positive integers chosen from $\{[G],\ [G]+1\}$ such that

$$\lim_{k\to\infty} (p_1 p_2 \cdots p_k)^{\frac{1}{k}} = G.$$

Set

$$\begin{aligned} q_1 &= p_1 p_2 \geq 2^2 = 2^1 + 2, \\ q_2 &= p_3 p_4 p_5 \geq 2^3 > 2^2 + 2, \\ q_3 &= p_6 p_7 p_8 p_9 \geq 2^4 > 2^3 + 2, \\ &\vdots \end{aligned}$$

In general, for $n = 1, 2, 3, \ldots$, we set

$$q_n = p_{\varphi(n-1)+1} \cdots p_{\varphi(n)} \geq 2^{n+1} \geq 2^n + 2,$$

where $\varphi(0) = 0$, and

$$\varphi(n) = 2 + 3 + \cdots + (n+1) = \frac{n(n+3)}{2}, \quad n = 1, 2, \ldots .$$

With the sequence $\{q_n\}$ in hand, we can proceed as in the proof of Lemma 2 and obtain a compact subset A of $[0,1]$ whose Hausdorff p-dimensional measure is

$$\begin{aligned} H_p(A) &= \lim_{l\to\infty} \frac{2 \cdot 2^2 \cdots 2^{2^l}}{(q_1 q_2 \cdots q_{2^l})^p} = \lim_{l\to\infty} \frac{2^{2^{l-1}(2^l+1)}}{(p_1 p_2 \cdots p_{\varphi(2^l)})^p} \\ &= \lim_{l\to\infty} \Big(\frac{2^{2^l+1/2^l+3}}{(p_1 p_2 \cdots p_{\varphi(2^l)})^{p/\varphi(2^l)}}\Big)^{\varphi(2^l)}. \end{aligned}$$

It is plain that $H_p(A) = \infty$ whenever $G^p < 2$, and $H_p(A) = 0$ whenever $G^p > 2$. Hence, $\dim A = r$. The rest of the proof goes exactly as in that of Lemma 2. □

The finite dimensional version of Lemmas 2 and 4 has been obtained earlier. It is, however, still open to us that if the upper bound $\frac{\log 2^n}{\log(2^n+2)}$ can be removed from the following statement.

Lemma 5 ([5, Theorem 2]; see also [7]). *Let $n \geq 2$ be any positive integer and $0 \leq r \leq 1$. There exists a continuous curve f from $[0,1]$ onto $[0,1]^n$ under which $[0,1]^n$ is the image of a compact set A of Hausdorff dimension r. Moreover, the restriction of f to A is one-to-one over the image of a dense subset provided $0 \leq r \leq \frac{\log 2^n}{\log(2^n+2)}$.*

Here comes the main result of this paper.

Theorem 6. *Let $0 \leq r \leq 1$ and M be a compact connected manifold of dimension n, where $n = 1, 2, \ldots, \omega$. We can construct a space-filling curve f of M under which the entire manifold M is the image of a compact subset A of $[0,1]$ of Hausdorff dimension r. Moreover, the restriction of f to A is one-to-one over the image of a dense subset provided $0 \leq r \leq \frac{\log 2^n}{\log(2^n+2)}$ ($= 1$ if $\dim M = \omega$).*

Proof. Suppose M is a compact, connected manifold of dimension n ($1 \leq n \leq \omega$). Then there exists a family of compact subsets $\{U_1, U_2, \ldots, U_m\}$ of M in which each U_i is homeomorphic to $[0,1]^n$, and $M \subseteq \bigcup_{i=1}^m \operatorname{int} U_i$. Without loss of generality, we can assume by connectedness of M that, $(U_1 \cup \cdots \cup U_k) \cap U_{k+1} \neq \emptyset$ for $k = 1, 2, \ldots, m-1$. There are homeomorphisms $h_1, h_2, \ldots, h_m$ from $U_1, U_2, \ldots, U_m$ onto $[0,1]^n$, and space-filling curves $g_1, g_2, \ldots, g_m$ from $[0, \frac{1}{2m-1}]$, $[\frac{2}{2m-1}, \frac{3}{2m-1}]$, ..., $[\frac{2m-2}{2m-1}, 1]$ onto $[0,1]^n$, respectively.

Suppose p_1 is a point in $U_1 \cap U_2$. Let

$$h_1^{-1}(\alpha_1, \alpha_2, \ldots) = p_1 = h_2^{-1}(\beta_1, \beta_2, \ldots),$$

where $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \beta_2, \ldots)$ are in $[0,1]^n$. Note that the surjective maps

$$f_1 = h_1^{-1} \circ g_1 : [0, \frac{1}{2m-1}] \to U_1 \quad \text{and} \quad f_2 = h_2^{-1} \circ g_2 : [\frac{2}{2m-1}, \frac{3}{2m-1}] \to U_2$$

are continuous. Let $(\alpha_1', \alpha_2', \ldots) = h_1(f_1(\frac{1}{2m-1})) = g_1(\frac{1}{2m-1})$ in $[0,1]^n$. Extend f_1 to $[0, \frac{3}{2(2m-1)}]$ by setting

$$f_1(\frac{1}{2m-1} + \lambda \frac{1}{2(2m-1)}) = h_1^{-1}(\lambda\alpha_1 + (1-\lambda)\alpha_1', \lambda\alpha_2 + (1-\lambda)\alpha_2', \ldots)$$

for $0 \le \lambda \le 1$. In particular,

$$f_1(\frac{3}{2(2m-1)}) = h_1^{-1}(\alpha_1, \alpha_2, \ldots) = p_1.$$

Similarly, let $(\beta_1', \beta_2', \ldots) = h_2(f_2(\frac{2}{2m-1})) = g_2(\frac{2}{2m-1})$ in $[0,1]^n$. Extend f_2 to $[\frac{3}{2(2m-1)}, \frac{3}{2m-1}]$ by setting

$$f_2(\frac{2}{2m-1} - \lambda \frac{1}{2(2m-1)}) = h_2^{-1}(\lambda\beta_1 + (1-\lambda)\beta_1', \lambda\beta_2 + (1-\lambda)\beta_2', \ldots)$$

for $0 \le \lambda \le 1$. In particular,

$$f_2(\frac{3}{2(2m-1)}) = h_2^{-1}(\beta_1, \beta_2, \ldots) = p_1.$$

Therefore, f_1 and f_2 agree at the point of the intersection of their domains. As a result, $f_1 \cup f_2$ is continuous from $[0, \frac{3}{2m-1}]$ onto $U_1 \cup U_2$.

In a similar manner, we can construct a continuous function $f = \bigcup_{k=1}^{m} f_k$ from $[0,1]$ onto M. Moreover, there are compact subsets B_k of $[\frac{2k-2}{2m-1}, \frac{2k-1}{2m-1}]$ as in Lemmas 2, 4 or 5 such that each B_k is of any pre-assigned Hausdorff dimension r, for $0 \le r \le 1$, and $g_k(B_k)$ fills up the whole of $[0,1]^n$. In case $0 \le r \le \frac{\log 2^n}{\log(2^n+2)}$, we can also assume that g_k is one-to-one over the image of a dense subset of B_k for each $k = 1, 2, \ldots, m$.

We set

$$\begin{aligned} A_1 =& B_1, \\ A_2 =& \overline{g_2^{-1}(h_2(U_2 \backslash U_1)) \cap B_2}, \\ &\vdots \\ A_n =& \overline{g_n^{-1}(h_n(U_n \backslash (U_1 \cup \cdots \cup U_{n-1}))) \cap B_n}. \end{aligned}$$

Since h_k is a homeomorphism, we see that each $C_k = g_k^{-1}(h_k(U_k \backslash (U_1 \cup \cdots \cup U_{k-1}))) \cap B_k$ is an open subset of B_k for $k = 1, 2, \ldots, n$. Set $A = \bigcup_{k=1}^{n} A_k \subseteq [0,1]$. Then A is a compact set of Hausdorff dimension r such that $f(A) = M$. Moreover, the restriction of f to A is one-to-one over the image of a dense subset of A contained in $\bigcup_{k=1}^{\infty} C_k$ provided $0 \le r \le \frac{\log 2^n}{\log(2^n+2)}$. □

3. Two examples

Example 7. A space-filling curve of the 3-dimensional cube $[0,1]^3$.

A space-filling curve $t \mapsto (x(t), y(t), z(t))$ of $[0,1]^3$ is given by writing

$$t = 0.t_1t_2\cdots \text{ in base 10 expansion}$$

and

$$\left.\begin{array}{l} x(t) = 0.x_1x_2\cdots \\ y(t) = 0.y_1y_2\cdots \\ z(t) = 0.z_1z_2\cdots \end{array}\right\} \text{ in base 2 expansion}$$

(in particular, $t_0 = x_0 = y_0 = 0$) such that for $k \geq 1$,

$$(x_k, y_k, z_k) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } 0 \leq t_k - 1 = 4\alpha + 2\beta + \gamma \leq 7; \\ (x_{k-1}, y_{k-1}, z_{k-1}), & \text{if } t_k = 0 = t_{k-1} \text{ or } t_k = 9 = t_{k-1}; \\ (1 - x_{k-1}, 1 - y_{k-1}, 1 - z_{k-1}), & \text{if } t_k = 0 \neq t_{k-1} \text{ or } t_k = 9 \neq t_{k-1}. \end{cases}$$

In general, the first k digits (in base 2) of $x(t)$, $y(t)$ and $z(t)$ can be calculated in terms of the first k digits (in base 10) of t. The image of

$$A = \left\{ \sum_{k=1}^{\infty} \frac{t_k}{10^k} : t_k = 1, 2, \ldots, 8,\ k = 1, 2, \ldots \right\}$$

fills up the entire cube $[0,1]^3$. In this case, $\dim A = \log 8/\log 10$ and f is one-to-one over the image of a dense subset of the compact set A.

To have an idea how the Hilbert cube $\mathbb{H} = \prod_{n=1}^{\infty} [0, \frac{1}{2^{n-1}}]$ is filled up, we re-scale our curve to the one $f(t) = (x(t), y(t)/2, z(t)/4)$. In Figure 1, we draw three polygons each of which approximates this space-filling curve within $1/2$ (order 1), $1/4$ (order 2), and $1/8$ (order 3) uniformly in all x-, y- and z-directions, respectively. They are obtained by making linear interpolation for the sets of data consisting of first one, two, and three digits of t, $x(t)$, $y(t)$ and $z(t)$, respectively, according to the methods described in [10] (in which we represent $1 = 0.99\cdots$ in base 10 for convenience).

Example 8. A space-filling curve of the ellipsoid $E = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ $(a, b, c > 0)$.

We first construct a space-filling curve of the sphere $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Let $0 < \epsilon < 1$ and

$$U_1 = \{(x,y,z) \in S^2 : -1 \leq z \leq \epsilon\}$$
$$U_2 = \{(x,y,z) \in S^2 : -\epsilon \leq z \leq 1\}.$$

Then $\{U_1, U_2\}$ is a compact covering of S. We are going to define the homeomorphisms h_i from U_i onto $[0,1]^2$ for $i = 1, 2$.

Consider the stereographic projections $P_i : U_i \longrightarrow D$ via the north pole (when $i = 1$) and the south pole (when $i = 2$), respectively, where

$$D = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 \leq \frac{1+\epsilon}{1-\epsilon}\}.$$

It is easy to see that

$$P_1(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z}) \quad \text{and} \quad P_2(x,y,z) = (\frac{x}{1+z}, \frac{y}{1+z}).$$

The next step is to consider the circle-to-square map

$$h'(a,b) = \begin{cases} \frac{\|(a,b)\|_2}{\|(a,b)\|_\infty}(a,b) & \text{if } (a,b) \neq (0,0), \\ (0,0) & \text{if } (a,b) = (0,0), \end{cases}$$

where

$$\|(a,b)\|_2 = \sqrt{a^2+b^2} \quad \text{and} \quad \|(a,b)\|_\infty = \max\{|a|,|b|\}.$$

It is plain that the map

$$h(a,b) = \frac{1}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}}h'(a,b) + (\frac{1}{2}, \frac{1}{2})$$

is a homeomorphism from D onto $[0,1]^2$. Consequently, $h_i = h \circ P_i$ is a homeomorphism from U_i onto $[0,1]^2$ for $i = 1, 2$.

Let $g : [0,1] \longrightarrow [0,1]^2$ be a space-filling curve. For instance, we can take g to be the one given by Lemma 5 as in [5, Example 3]. More precisely, the space-filling curve $g(t) = (x(t), y(t))$ is given by writing

$$t = 0.t_1t_2\cdots \text{ in base 6 expansion}$$

and

$$\left.\begin{array}{l} x(t) = 0.x_1x_2\cdots \\ y(t) = 0.y_1y_2\cdots \end{array}\right\} \text{ in base 2 expansion}$$

(in particular, $t_0 = x_0 = y_0 = 0$) such that for $k \geq 1$,

$$(x_k, y_k) = \begin{cases} (\alpha, \beta), & \text{if } 0 \leq t_k - 1 = 2\alpha + \beta \leq 3; \\ (x_{k-1}, y_{k-1}), & \text{if } t_k = 0 = t_{k-1} \text{ or } t_k = 5 = t_{k-1}; \\ (1 - x_{k-1}, 1 - y_{k-1}), & \text{if } t_k = 0 \neq t_{k-1} \text{ or } t_k = 5 \neq t_{k-1}. \end{cases}$$

Then $g(A) = [0,1]^2$ for the compact set $A = \left\{\sum_{k=1}^\infty \frac{t_k}{6^k} : t_k = 1,2,3,4,\ k = 1,2,\dots\right\}$ of Hausdorff dimension $\log 4/\log 6$. Moreover, g is one-to-one over the image of a dense subset of A.

Let

$$f_1 : [0, 1/3] \to U_1 \text{ and } f_2 : [2/3, 1] \to U_2$$

be defined by

$$f_1(t) = h_1^{-1}(g(3t)) \text{ and } f_2(t) = h_2^{-1}(g(3-3t)).$$

Following the proof of Theorem 6, we observe that

$$h_1f_1(1/3) = h_2f_2(2/3) = g(1) = (1,1) \in [0,1]^2,$$

$$(\sqrt{1/2}, \sqrt{1/2}, 0) \in U_1 \cap U_2,$$

and

$$h_1(\sqrt{1/2}, \sqrt{1/2}, 0) = h_2(\sqrt{1/2}, \sqrt{1/2}, 0) = (\frac{1}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}} + \frac{1}{2}, \frac{1}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}} + \frac{1}{2}) \in [0,1]^2.$$

We can extend f_1 from $[0,1/3]$ to $[0,1/2]$ and f_2 from $[2/3,1]$ to $[1/2,1]$ by setting

$$\begin{aligned} f_1(\frac{1}{3} + \lambda\frac{1}{6}) &= h_1^{-1}((1-\lambda)\cdot 1 + \lambda\cdot(\frac{1}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}} + \frac{1}{2}), (1-\lambda)\cdot 1 + \lambda\cdot(\frac{1}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}} + \frac{1}{2})) \\ &= h_1^{-1}(1 - \frac{\lambda}{2} + \frac{\lambda}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}}, 1 - \frac{\lambda}{2} + \frac{\lambda}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}}), \end{aligned}$$

and similarly,

$$f_2(\frac{2}{3} - \lambda\frac{1}{6}) = h_2^{-1}(1 - \frac{\lambda}{2} + \frac{\lambda}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}}, 1 - \frac{\lambda}{2} + \frac{\lambda}{2}\sqrt{\frac{1-\epsilon}{1+\epsilon}})$$

for $0 \leq \lambda \leq 1$. In this way, $f_1(1/2) = f_2(1/2) = (\sqrt{1/2}, \sqrt{1/2}, 0)$ and we have a continuous map $f = f_1 \cup f_2$ from $[0, 1]$ onto S. Suppose

$$f(t) = (x(t), y(t), z(t)) \text{ for } t \in [0, 1].$$

Then, the map

$$g(t) = (ax(t), by(t), cz(t))$$

is a space-filling curve of the ellipsoid E. Moreover, g maps the $\frac{\log 4}{\log 6}$–dimensional compact set A onto E such that g is one-to-one over the image of a dense subset of A.

In Figure 2, we draw approximating polygons of g when $a = 2b = 4c = 1$ and $\epsilon = 0$ for demonstration. To make the picture more easily to be visualized, only the lower hemi-ellipsoid is shown. Note that setting $\epsilon = 0$ (for simplicity) in this case is still good enough for our task (either by direct observation or arguing by uniformity).

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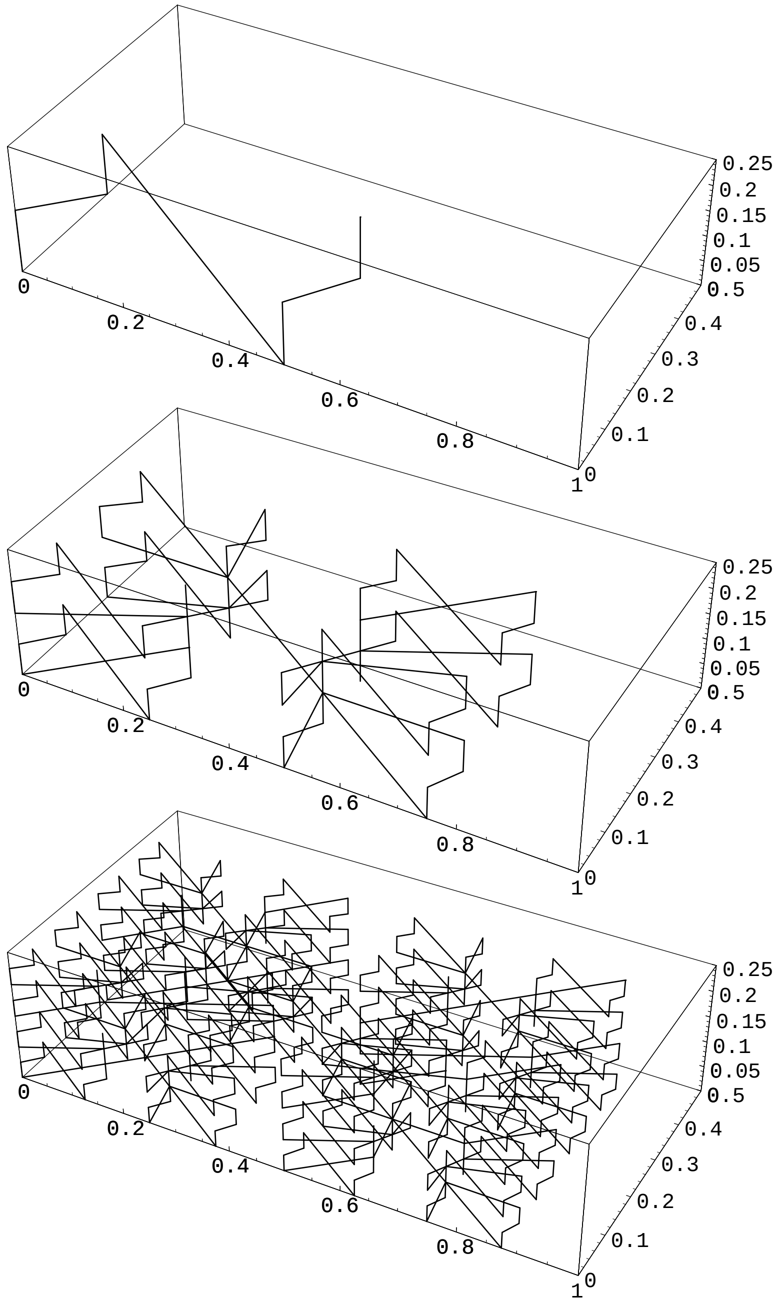

FIGURE 1. Approximating polygons of order 1 (top), 2 (middle), and 3 (bottom) of a space-filling curve of $[0,1] \times [0, \frac{1}{2}] \times [0, \frac{1}{4}]$. These figures are generated by Mathematica version 3.0 in SUN SPARC20-712.

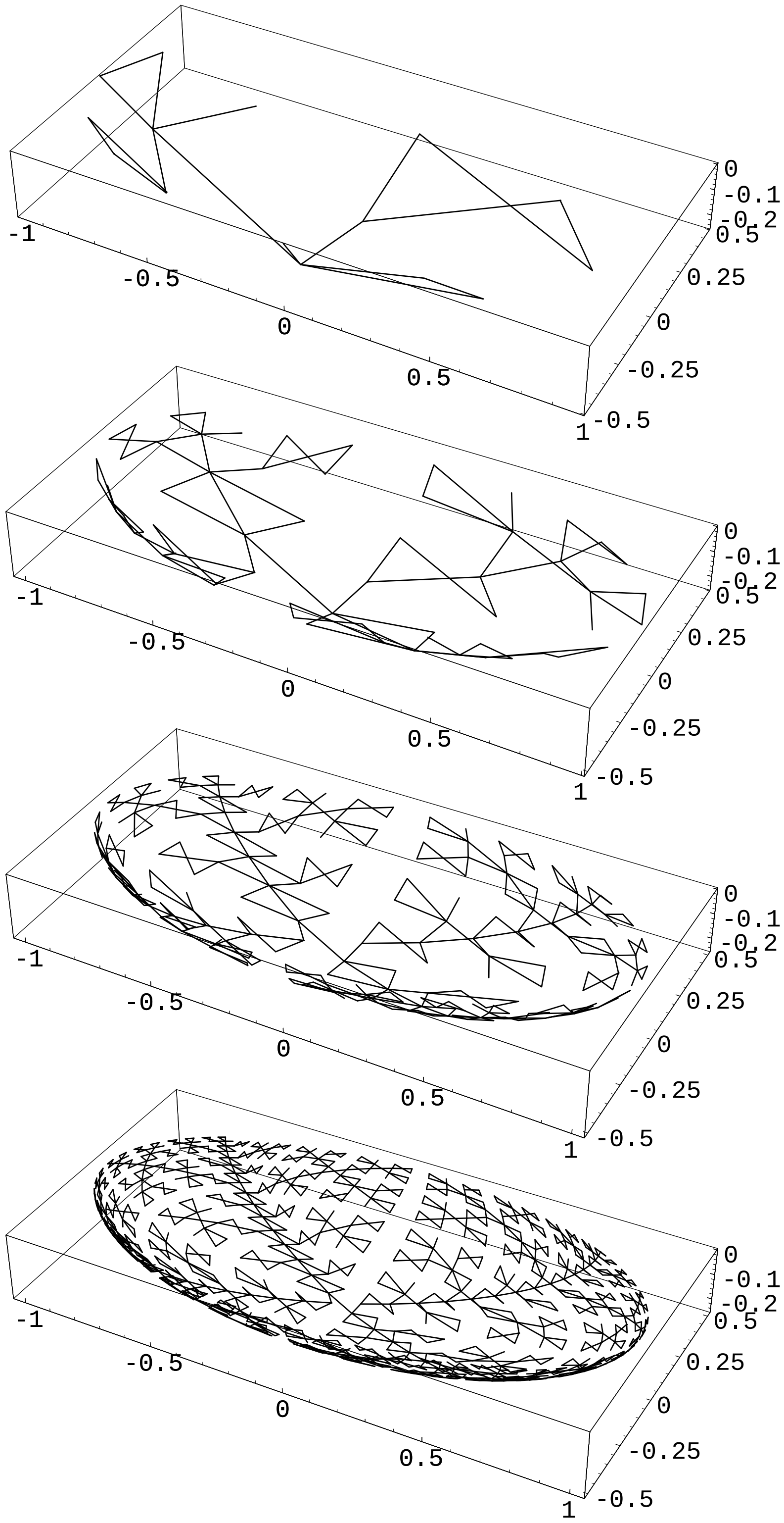

FIGURE 2. Approximating polygons of order 2 (top), 3 (second), 4 (third), and 5 (bottom) of the lower half of a space-filling curve of the ellipsoid $x^2 + 4y^2 + 16z^2 = 1$. These figures are generated by Mathematica version 3.0 in SUN SPARC20-712.