SOME CONVERSES OF THE STRONG SEPARATION THEOREM *

Hwa-Long Gau and Ngai-Ching Wong
Department of Applied Mathematics
National Sun Yat-sen University
Kao-hsiung, 80424, Taiwan, R.O.C.

Abstract

A convex subset B of a real locally convex space X is said to have the *separation property* if it can be separated from any closed convex subset A of X, which is disjoint from B, by a closed hyperplane. The strong separation theorem says that if B is weakly compact then it has the separation property. In this paper, we present several versions for the converse and discuss some applications. For example, we prove that a normed space is reflexive if and only if its closed unit ball has the separation property. Results in this paper can be considered as generalizations and supplements of the famous James' Theorem.

1 Introduction

Definition. Let B be a bounded convex subset of a real locally convex (Hausdorff) space X. B is said to have the *separation property* if it can be strictly separated from any closed convex subset A of X, which is disjoint from B, by a closed hyperplane, *i.e.* there is a continuous linear functional f of X such that

$$\inf\{f(x) : x \in A\} > \sup\{f(x) : x \in B\}.$$

B is said to have *James' property* if every continuous linear functional f of X attains its supremum on B, *i.e.* there is a b in B such that

$$f(b) = \sup\{f(x) : x \in B\}.$$

*This research is partially supported by National Science Council of Taiwan, R.O.C.

The classical strong separation theorem of Klee [7] states that if B is weakly compact then B has the separation property. It is also plain that if B has the separation property then B has James' property. In this paper, we shall investigate possible converses of the above two implications.

In a series of papers [2–6], R.C. James established the famous James' Theorem which was extended by J. D. Pryce [8] as

James' Theorem. *For a complete bounded convex subset B of a locally convex space X, B is weakly compact if and only if B has James' property.*

A major application of James' Theorem is a characterization of the reflexivity of Banach spaces: A Banach space E is reflexive if and only if the closed unit ball of E has James' property.

James' Theorem cannot be extended further to incomplete bounded convex sets. In [5], R.C. James presented a counter example to show that a bounded convex set with James' property is not necessarily weakly compact even if it is the closed unit ball of a *normed* space. We shall use the same example to show that a bounded convex set with James' property does not necessarily have the separation property, either (see Example 6). In other words, the separation property is closer to weak compactness than James' property in general. As an evidence, we obtain

Theorem 1. *A bounded convex body (i.e. convex set with nonempty interior) B in a real normed space X is weakly compact if and only if B has the separation property. In particular, a real normed space is reflexive if and only if its closed unit ball has the separation property.*

Conjecture. A bounded convex subset B of a real locally convex space X is weakly compact if and only if B has the separation property.

We shall present in Theorem 3 a sufficient condition under which our conjecture holds. Theorem 7 demonstrates an application of our results. It shows clearly that even a partial answer of our conjecture can quite improve many classical results, in particular, for those involving completeness conditions. Although we discuss only real locally convex spaces in this paper, our results should be easily extended to complex cases.

We would like to take this opportunity to express our great gratitude to Louis de Branges for his kind motivation and encouragement, and Yau-chuen Wong and the referee for their valuable suggestions.

2 Main results

In the following, B always denotes a bounded convex subset of a real locally convex space X. We note that if B has the separation property then B is weakly closed. It is clear that Theorem 1 is a corollary of James' Theorem and the following lemma.

Lemma 2. *Let B be a bounded convex body in a real normed space X. If B has the separation property then B is complete.*

PROOF. W.L.O.G. we can assume $0 \in B$. Let $\tilde{B}$ be the closure of B in the completion $\tilde{X}$ of X. Note that $\tilde{B}$ is a bounded convex body in $\tilde{X}$. For any nonzero b in $\tilde{B}$, λb belongs to the boundary of $\tilde{B}$, where $\lambda = \sup\{k : kb \in \tilde{B}\} \geq 1$. We want to show that $\tilde{B} = B$. It suffices to verify that B contains the boundary of $\tilde{B}$.

Suppose there were an element b in the boundary of $\tilde{B}$ such that $b \notin B$. Let b be contained in a supporting hyperplane H of $\tilde{B}$ such that $H = \{x \in \tilde{X} : f(x) = 1\}$ and $\tilde{B} \subset \{x \in \tilde{X} : f(x) \leq 1\}$ for some continuous linear functional f of $\tilde{X}$. In particular, $f(b) = 1$. Let $b_n = (1 + \frac{1}{n})b$ for $n = 1, 2, 3, \cdots$. Let $B_{\tilde{X}}(a; \delta)$ denote the open ball $\{x \in \tilde{X} : \|x - a\| < \delta\}$. Since $B_{\tilde{X}}(b_n; \frac{1}{n}) \cap \{x \in \tilde{X} : f(x) > 1 + \frac{1}{n}\}$ is non-empty and open in $\tilde{X}$ for each $n = 1, 2, \cdots$, and X is dense in $\tilde{X}$, we can choose a_n's from X so that $a_n \in B_{\tilde{X}}(b_n; \frac{1}{n}) \cap \{x \in \tilde{X} : f(x) > 1 + \frac{1}{n}\}$. Then $f(a_n) > 1 + \frac{1}{n}$ for $n = 1, 2, 3, \cdots$, and the sequence a_n's converges to b in norm.

Let A be the closed convex hull of a_n's in X. We want to show that $A \cap B = \emptyset$. Suppose an element y in X exists such that $y \in A \cap B$. Let N be a positive integer such that $B_{\tilde{X}}(b; \frac{2}{N}) \cap B_{\tilde{X}}(y; \frac{2}{N}) = \emptyset$. Since $y \in A$, there exists a sequence y_n's of convex combinations of a_n's converges to y in norm. For each n, write $y_n = \sum_{i=1}^{k_n} \alpha_i^n a_i$, where $\alpha_i^n \geq 0$ for $i = 1, 2, \cdots, k_n$, $\sum_{i=1}^{k_n} \alpha_i^n = 1$ and k_n is a positive integer depending on n. Since $y_n \to y$ in norm and $f(y) = 1$, there exists a positive integer M_1 such that $f(y_n) < 1 + \frac{1}{N}$ for all $n \geq M_1$. For each $n \geq M_1$, $1 + \frac{1}{N} > f(\sum_{i=1}^{k_n} \alpha_i^n a_i) = \sum_{i=1}^{k_n} \alpha_i^n f(a_i) > \sum_{i=1}^{k_n} \alpha_i^n (1 + \frac{1}{i}) = 1 + \sum_{i=1}^{k_n} \frac{\alpha_i^n}{i}$. This implies $\sum_{i=1}^{k_n} \frac{\alpha_i^n}{i} < \frac{1}{N}$. On the other hand, there exists a positive integer M_2 such that $y_n \in B_{\tilde{X}}(y; \frac{2}{N}), \forall n \geq M_2$. For $n \geq M = \max\{M_1, M_2\}$, $\|y_n - b\| = \|\sum_{i=1}^{k_n} \alpha_i^n a_i - b\| = \|\sum_{i=1}^{k_n} \alpha_i^n (a_i - b)\| \leq \sum_{i=1}^{k_n} \alpha_i^n \|a_i - b\| < 2\sum_{i=1}^{k_n} \frac{\alpha_i^n}{i} < \frac{2}{N}$. This implies $y_n \in B_{\tilde{X}}(b; \frac{2}{N}), \forall n \geq M$. This contradicts to the fact that $B_{\tilde{X}}(b; \frac{2}{N}) \cap B_{\tilde{X}}(y; \frac{2}{N}) = \emptyset$. Hence $A \cap B = \emptyset$.

By the separation property of B, there is a continuous linear functional g of X such that

$$\sup\{g(u) : u \in B\} < \inf\{g(a) : a \in A\}.$$

Let g' be the continuous extension of g to $\tilde{X}$. Since $a_n \to b$ as $n \to \infty$ in $\tilde{X}$,

$$g'(b) = \lim_{n\to\infty} g(a_n) \geq \inf\{g(a) : a \in A\} > \sup\{g(u) : u \in B\} \geq g'(b).$$

This is a contradiction! Therefore $B = \tilde{B}$, and thus B is complete. □

Let $(X, \Im)$ be a locally convex space. A subset B of X is said to be *absolutely convex* if $\lambda a + \beta b \in B$ whenever $a, b \in B$ and $|\lambda| + |\beta| \leq 1$. For any absolutely convex $\Im$-bounded subset B of X, let $X(B)$ be the linear span of B. Then $X(B) = \bigcup_n nB$, and B is absorbing in $X(B)$. Hence the gauge γ_B of B, defined by

$$\gamma_B(x) = \inf\{\lambda > 0 : x \in \lambda B\},$$

is a seminorm on $X(B)$ and

$$\{x \in X(B) : \gamma_B(x) < 1\} \subset B \subset \{x \in X(B) : \gamma_B(x) \leq 1\}.$$

Moreover, the boundedness of B ensures that $\Im|_{X(B)}$ (the relative topology induced by $\Im$) is coarser than the $\gamma_B(\cdot)$-topology. Thus, γ_B is actually a norm on $X(B)$. We write $\overline{B}^{\gamma_B}$ for the closure of B in $(X(B), \gamma_B)$, and $\overline{B}^{\Im|_{X(B)}}$ for the closure of B in $(X(B), \Im|_{X(B)})$.

Theorem 3. *Let B be a bounded absolutely convex subset of a real locally convex space $(X, \Im)$ such that $(X(B), \gamma_B)^* = (X(B), \Im|_{X(B)})^*$. Then B is weakly compact if and only if B has the separation property.*

Proof. The necessity is clear. For the sufficiency, we shall show that for any closed and bounded convex subset A of $(X(B), \gamma_B)$, which is disjoint from B, can be strictly separated from the closed unit ball B of $(X(B), \gamma_B)$. Since the $\gamma_B(\cdot)-$topology is consistent with the duality $\langle (X(B), \Im|_{X(B)}), (X(B), \Im|_{X(B)})^* \rangle$, A is also a closed convex subset of

$(X(B), \Im|_{X(B)})$. By the boundedness of A in $(X(B), \gamma_B)$, A is closed in $(X, \Im)$. The separation property of B provides an f in $(X, \Im)^*$ such that

$$\sup\{f(b) : b \in B\} < \inf\{f(a) : a \in A\}.$$

Let $g = f|_{X(B)}$, then $g \in (X(B), \Im|_{X(B)})^* = (X(B), \gamma_B)^*$ and

$$\sup\{g(b) : b \in B\} < \inf\{g(a) : a \in A\}.$$

Therefore, the closed unit ball B of $(X(B), \gamma_B)$ has the separation property, too. By Theorem 1, B is weakly compact in $(X(B), \gamma_B)$. Note that the topology $\Im|_{X(B)}$ is coarser than the $\gamma_B(\cdot)$- topology. It turns out that B is weakly compact in $(X, \Im)$ and we complete the proof. □

Remark: The following two examples indicate that the weak compactness of B and the condition that $(X(B), \gamma_B)^* = (X(B), \Im|_{X(B)})^*$ in last theorem are independent in general.

Example 4. Let B be the closed unit ball of the reflexive Banach space $(\ell_2, \|\cdot\|_2)$. Let $(X, \Im) = (\ell_2, \|\cdot\|_\infty)$. Then $(X(B), \gamma_B) = (\ell_2, \|\cdot\|_2)$ and B is weakly compact in $(X(B), \gamma_B)$. Since the $\|\cdot\|_\infty$−topology is coarser than the $\|\cdot\|_2$−topology, B is weakly compact in $(X, \Im)$. But

$$(X(B), \gamma_B)^* = (\ell_2, \|\cdot\|_2)^* \neq (\ell_2, \|\cdot\|_\infty)^* = (X(B), \Im|_{X(B)})^*.$$

□

Example 5. Let $X = \ell_0$, the space of finite sequences, and B be the closed unit ball of the normed space $(\ell_0, \|\cdot\|_\infty)$. Let $\Im$ be the weak topology of $(\ell_0, \|\cdot\|_\infty)$. Then

$$(X(B), \gamma_B)^* = (\ell_0, \|\cdot\|_\infty)^* = (\ell_0, \Im)^* = (X(B), \Im|_{X(B)})^*.$$

But B is not weakly compact in $(X, \Im)$, since $(\ell_0, \|\cdot\|_\infty)$ is not reflexive. □

3 A counter example

The following example is based on a construction of R.C. James [5].

Example 6. Let E be a countable real Hilbert product of increasing finite dimensional c_0-spaces, so that the members of E are of type $x = (x_1^1; x_1^2, x_2^2; x_1^3, x_2^3, x_3^3; \cdots)$ with

$$\|x\| = [| x_1^1 |^2 + (\sup\{| x_1^2 |, | x_2^2 |\})^2 + (\sup\{| x_1^3 |, | x_2^3 |, | x_3^3 |\})^2 + \cdots]^{1/2} < \infty. \quad (1)$$

Let X be the linear span of all members x of E such that

$$| x_1^n |=| x_2^n |= \cdots =| x_n^n | \qquad \text{for all } n = 1, 2, \cdots. \tag{2}$$

Since E is a Hilbert product of reflexive spaces, E is reflexive. It is easy to see that X is dense in E. Note that

(*) If $x \in X$ and x is a linear combination of n members of X that satisfying (2), then for each $m > 2^n$ at least two of $x_1^m, \cdots, x_m^m$ are equal.

Thus the sequence $\frac{1}{n}$'s belongs to E but not to X. Therefore $X \neq E$ and X is not complete. In particular, the closed unit ball B of X is not weakly compact.

We shall verify two facts:

(a) B has James' property (this part is due to R.C. James [5]).

Let f be an arbitrary continuous linear functional on E and x in E be such that $\|x\| = 1$ and $f(x) = \|f\|$. Then there is a sequence of numbers (f_i^n) such that

$$f(x) = f_1^1 x_1^1 + (f_1^2 x_1^2 + f_2^2 x_2^2) + (f_1^3 x_1^3 + f_2^3 x_2^3 + f_3^3 x_3^3) + \cdots. \tag{3}$$

The norm of x as given by (1) is not changed if for each n we replace each x_i^n by $\pm \sup_i |x_i^n|$, where the "+" is used if $f_i^n \geq 0$ and the "−" if $f_i^n < 0$. The changes do not decrease the sum in (3), so the sum does not change and the new x is a member of the closed unit ball of X at which f attains its supremum.

(b) B does *not* have the separation property.

Let

$$x = \left(\frac{\sqrt{3}}{2^{n+j-1}}\right)_{j=1,\cdots,n}^{n=1,2,3,\cdots} = \sqrt{3}\left(\frac{1}{2}; \frac{1}{2^2}, \frac{1}{2^3}; \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}; \cdots; \frac{1}{2^n}, \frac{1}{2^{n+1}}, \cdots, \frac{1}{2^{2n-1}}; \cdots\right).$$

By (*), $x \notin X$ and $\|x\| = 1$. Let

$$x_1 = \sqrt{3}(\tfrac{1}{2}; \tfrac{1}{2^2}, \tfrac{1}{2^2}; \tfrac{1}{2^3}, \tfrac{1}{2^3}, \tfrac{1}{2^3}; \cdots; \tfrac{1}{2^n}, \cdots, \tfrac{1}{2^n}; \cdots).$$
$$x_2 = \sqrt{3}(\tfrac{1}{2}; \tfrac{1}{2^2}, \tfrac{1}{2^3}; \tfrac{1}{2^3}, \tfrac{1}{2^3}, \tfrac{1}{2^3}; \cdots; \tfrac{1}{2^n}, \cdots, \tfrac{1}{2^n}; \cdots).$$
$$x_3 = \sqrt{3}(\tfrac{1}{2}; \tfrac{1}{2^2}, \tfrac{1}{2^3}; \tfrac{1}{2^3}, \tfrac{1}{2^4}, \tfrac{1}{2^5}; \tfrac{1}{2^4}, \tfrac{1}{2^4}, \tfrac{1}{2^4}, \tfrac{1}{2^4}; \cdots; \tfrac{1}{2^n}, \cdots, \tfrac{1}{2^n}; \cdots).$$
$$\vdots$$
$$x_n = \sqrt{3}(\tfrac{1}{2}; \tfrac{1}{2^2}, \tfrac{1}{2^3}; \tfrac{1}{2^3}, \tfrac{1}{2^4}, \tfrac{1}{2^5}; \cdots; \tfrac{1}{2^n}, \tfrac{1}{2^{n+1}}, \cdots, \tfrac{1}{2^{2n-1}}; \tfrac{1}{2^{n+1}}, \tfrac{1}{2^{n+1}}, \cdots, \tfrac{1}{2^{n+1}}; \cdots).$$
$$\vdots$$

It is easy to see that $x_n \in X$ and $\|x_n\| = 1$ for all $n = 1, 2, \cdots$ and $x_n \to x$ in norm as $n \to \infty$. Let $a_n = (1 + \frac{1}{n})x_n$. It follows that $a_n \in X$, $\|a_n\| = 1 + \frac{1}{n}$ for all $n = 1, 2, \cdots$, and $a_n \to x$ in norm as $n \to \infty$.

Let A be the closed convex hull of a_n's in X. We want to verify that A and the closed unit ball B of X are disjoint. Suppose $y = (y_i^m)_{i=1,\cdots,n}^{m=1,2,\cdots}$ is an element of the convex hull of a_n's. Let $y = \sum_{i=1}^k \alpha_i a_{n_i}$ for some positive integer k, where $\alpha_i \geq 0$ and $\sum_{i=1}^k \alpha_i = 1$. Then $\|y\| = 1 + p$ where $p = \sum_{i=1}^k \frac{\alpha_i}{n_i} > 0$, and $y_1^m > y_2^m > \cdots > y_m^m$ for $m = 1, \cdots, n_k$ (W.L.O.G. assume $n_1 < n_2 < \cdots < n_k$). In particular, the convex hull of a_n's is disjoint from B. If a is a cluster point of the convex hull of a_n's in E with $\|a\| = 1$ then $a_1^m > a_2^m > \cdots > a_m^m$ for all m. In fact, there exists a sequence y_n's in the convex hull of a_n's such that $y_n \to a$ as $n \to \infty$, say $y_n = \sum_{i=1}^{k_n} \alpha_i^n a_i = \sum_{i=1}^{k_n} \alpha_i^n (1 + \frac{1}{i}) x_i$ where $\alpha_i^n \geq 0$, $\sum_{i=1}^{k_n} \alpha_i^n = 1$ and k_n is a positive integer. Then for any positive integer m, $1 \leq j \leq m$,

$$(y_n)_j^m = \frac{\sqrt{3}}{2^m}(\sum_{i=1}^{m-1} \alpha_i^n + \sum_{i=1}^{m-1} \frac{\alpha_i^n}{i}) + \frac{\sqrt{3}}{2^{m+j-1}}(\sum_{i=m}^{k_n} \alpha_i^n + \sum_{i=m}^{k_n} \frac{\alpha_i^n}{i}).$$

Now, for any positive integer m, $1 \leq j \leq m - 1$,

$$\begin{aligned} a_j^m - a_{j+1}^m &= \lim_{n\to\infty} [(y_n)_j^m - (y_n)_{j+1}^m] \\ &= (\frac{\sqrt{3}}{2^{m+j-1}} - \frac{\sqrt{3}}{2^{m+j}})[\lim_{n\to\infty} (\sum_{i=m}^{k_n} \alpha_i^n + \sum_{i=m}^{k_n} \frac{\alpha_i^n}{i})] \\ &\geq (\frac{\sqrt{3}}{2^{m+j-1}} - \frac{\sqrt{3}}{2^{m+j}})(\lim_{n\to\infty} \sum_{i=m}^{k_n} \alpha_i^n). \end{aligned}$$

If $\lim_{n\to\infty} \sum_{i=m}^{k_n} \alpha_i^n = 0$ then for any given $\varepsilon > 0$, there exists a positive integer M such that $\sum_{i=m}^{k_n} \alpha_i^n < \varepsilon$ for all $n \geq M$. This implies $\sum_{i=1}^m \alpha_{i=1}^n > 1 - \varepsilon$ for all $n \geq M$. It follows that for all $n \geq M$,

$$(y_n)_j^m \geq \frac{\sqrt{3}}{2^m}(\sum_{i=1}^{m-1} \alpha_i^n + \sum_{i=1}^{m-1} \frac{\alpha_i^n}{i}) \geq \frac{\sqrt{3}}{2^m}(1 + \frac{1}{m})(\sum_{i=1}^{m-1} \alpha_i^n) > \frac{\sqrt{3}}{2^m}(1 + \frac{1}{m})(1 - \varepsilon).$$

Therefore

$$a_j^m = \lim_{n\to\infty} (y_n)_j^m \geq \frac{\sqrt{3}}{2^m}(1 + \frac{1}{m})(1 - \varepsilon).$$

Let $\varepsilon \to 0$, we have $a_j^m \geq \frac{\sqrt{3}}{2^m}(1 + \frac{1}{m})$ for all positive integer m and $1 \leq j \leq m$. Then $\|a\| > 1$, which contradicts to the fact that $\|a\| = 1$. Hence $\lim_{n\to\infty} \sum_{i=m}^{k_n} \alpha_i^n > 0$ and

$$a_j^m - a_{j+1}^m \geq (\frac{\sqrt{3}}{2^{m+j-1}} - \frac{\sqrt{3}}{2^{m+j}})(\lim_{n\to\infty} \sum_{i=m}^{k_n} \alpha_i^n) > 0$$

for all positive integer m and $1 \leq j \leq m-1$. By (*), $a \notin X$ and consequently, $a \notin B \subset X$. Hence A and B are disjoint. Next, we show that A and B cannot be strictly separated. Suppose there were a continuous linear functional f of E such that

$$\sup\{f(b) : b \in B\} = \|f\| < \inf\{f(a); a \in A\}.$$

Since $a_n \to x$ as $n \to \infty$,

$$\inf\{f(a); a \in A\} \leq \lim_{n\to\infty} f(a_n) = f(x) \leq \|f\|.$$

This is a contradiction! Hence A and B cannot be strictly separated. □

4 Applications

Let us recall the classical theorem that a Banach space is reflexive if and only if its unit ball is weakly sequentially compact [1]. The following extends some James' results from Banach spaces to normed spaces, cf. [4].

Theorem 7. *Let B be the closed unit ball of a real normed space N. Then the following are equivalent:*

(1) *B is weakly compact.*

(2) *B is weakly countably compact.*

(3) *For each sequence x_n's in B there is an x in B such that for all continuous linear functionals f,*

$$\underline{\lim} f(x_n) \leq f(x) \leq \overline{\lim} f(x_n).$$

(4) *If K_n's is a decreasing sequence of closed convex sets in X and $B \cap K_n$ is non-empty for each n, then $B \cap (\cap_{n\geq 1} K_n)$ is non-empty.*

(5) *B is weakly sequentially compact.*

(6) *If S is a weakly closed set and $B \cap S$ is empty, then $d(B, S) = \inf\{\|b - s\| : b \in B, s \in S\} > 0$.*

(7) *B has the separation property.*

PROOF. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are proved in [4], and the implication (7) $\Rightarrow$ (1) follows from Theorem 1.

We shall show that (4) $\Rightarrow$ (7). Suppose (4) holds but there were a closed convex set A disjoint from B which cannot be strictly separated from B by a closed hyperplane. In particular, $d(A,B)=0$. We can thus choose a_n's in A and b_n's in B such that $\|a_n-b_n\|\to 0$ as $n\to\infty$. Let K_n be the closed convex hull of $\{b_n, b_{n+1}, \cdots\}$ for $n=1,2,\cdots$. We want to show that $\cap_{n\geq 1}K_n=\emptyset$. If there exists an element b in $\cap_{n\geq 1}K_n$, then for all continuous linear functionals f, we have

$$\underline{\lim} f(b_n)\leq f(b)\leq \overline{\lim} f(b_n).$$

As

$$|f(a_n)-f(b_n)|\leq \|f\|\|a_n-b_n\|\to 0,$$

we have

$$\underline{\lim} f(a_n)\leq f(b)\leq \overline{\lim} f(a_n). \quad \forall f\in X^*.$$

By the strong separation theorem, b is in the closed convex hull of $\{a_n, a_{n+1}, \cdots\}$ for $n=1,2,\cdots$. Then $b\in A\cap B$. This is a contradiction and thus $\cap_{n\geq 1}K_n=\emptyset$. This again conflicts with (4). Hence B has the separation property. □

We end this paper with an open problem which seems to be an intermediate (and possibly critical) step to our conjecture.

Problem. Does the continuous linear image of a bounded convex set with the separation property still have the separation property?

It is clear that similar questions concerning weak compactness and James' property have positive answers.

References

[1] W. F. Eberlein, *Weak compactness in Banach spaces,* Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51-53.

[2] R. C. James, *Weakly compact sets,* Trans. Amer. Math. Soc. **113** (1964), 129-140.

[3] R. C. James, *Weak compactness and separation*, Canad. J. Math. **16** (1964), 204-206.

[4] R. C. James, *Weakly compactness and reflexivity*, Israel J. Math. **2** (1964), 101-119.

[5] R. C. James, *A counterexample for a sup theorem in normed spaces*, Israel J. Math. **9** (1971), 511-512.

[6] R. C. James, *Reflexivity and the sup of linear functionals*, Israel J. Math. **13** (1972), 289-300.

[7] V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J., **18** (1951), 443-466, 875-883.

[8] J. D. Pryce, *Weak compactness in locally convex spaces,* Proc. Amer. Math. Soc. **17** (1966), 148-155.

[9] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1986.