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Mappings preserving zero products

by

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Abstract. Let $\theta : \mathcal{M} \to \mathcal{N}$ be a zero-product preserving linear map between algebras. We show that under some mild conditions θ is a product of a central element and an algebra homomorphism. Our result applies to matrix algebras, standard operator algebras, C^*-algebras and W^*-algebras.

1. Introduction. Let $\mathcal{M}$ and $\mathcal{N}$ be algebras over a field $\mathbb{F}$ and $\theta : \mathcal{M} \to \mathcal{N}$ a linear map. We say that θ is a *zero-product preserving* map if $\theta(a)\theta(b) = 0$ in $\mathcal{N}$ whenever $ab = 0$ in $\mathcal{M}$. For example, if h is an element in the center of $\mathcal{N}$ and $\varphi : \mathcal{M} \to \mathcal{N}$ is an algebra homomorphism then $\theta = h\varphi$ is zero-product preserving. In this paper, we show that in many interesting cases zero-product preserving linear maps arise in this way.

Let $C(X)$ be the algebra of all continuous complex functions defined on a compact Hausdorff space X. Then $C(X)$ bears several different structures: a commutative ring, a Banach algebra, a Banach lattice, a C^*-algebra and so on. We are interested in relations among these structures. For example, let $\theta : C(X) \to C(Y)$ be a linear map. Then θ is a ring isomorphism if and only if $\theta(f) = f \circ \sigma$ for a homeomorphism σ from Y onto X (see, e.g., [17, p. 57]). θ is a surjective isometry if and only if $\theta(f) = hf \circ \sigma$ for a homeomorphism σ from Y onto X and a unimodular continuous function h by the Banach–Stone Theorem (see, e.g., [21]). In other words, $\theta = h\varphi$, a product of h and an algebra isomorphism φ. On the other hand, θ is a lattice isomorphism if and only if $\theta(f) = hf \circ \sigma$ for a homeomorphism σ and a nonvanishing positive continuous function h (see, e.g., [1]). The common part of θ being a ring isomorphism, a surjective isometry and a lattice isomorphism is that θ preserves zero products. In fact, θ preserves zero products if and only if $\theta(f) = hf \circ \sigma$, where h can be zero somewhere and σ is a general continuous map (see, e.g., [21]). In [1, 2, 12, 16, 18, 20, 21],

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to name a few, the above relations are extended to the case where θ is not bijective, or is not continuous, or maps between vector-valued function spaces. We are interested in the question if the zero-product preserving property still plays an important role in a general algebraic setting.

Let $\theta : \mathcal{M} \to \mathcal{N}$ be a continuous zero-product preserving linear map between topological algebras. We see that in many situations, if $\mathcal{M}$ is unital then

$$\theta(1)\theta(ab) = \theta(a)\theta(b) \quad \text{for all } a, b \in \mathcal{M}. \tag{1.1}$$

In particular, $\theta(1)$ commutes with $\theta(a)$ for all a in $\mathcal{M}$. A direct consequence of (1.1) is that θ preserves commutativity. Tools provided in, for example, [6–10, 26, 27, 31] can thus be used.

In Section 2, we see that (1.1) holds when the subalgebra of $\mathcal{M}$ generated by its idempotents is dense in $\mathcal{M}$. If $\theta(1)$ is invertible in $\mathcal{N}$ or the subalgebra of $\mathcal{N}$ generated by $\theta(\mathcal{M})$ has an identity, then $\theta = \theta(1)\varphi$ for an algebra homomorphism φ. As an application, we give a rather complete description of all zero-product preserving linear maps between matrix algebras. Another nice case is when $\mathcal{M} = B(H)$ consists of all bounded linear operators on an infinite-dimensional complex Hilbert space H. In this case, it suffices to assume θ to be just additive and the conclusion $\theta = \theta(1)\varphi$ is still valid with φ being a ring homomorphism.

In Section 3, we show that (1.1) holds when $\mathcal{M}$ has a complete system of matrix units and the range of θ is dense in a prime subalgebra of $\mathcal{N}$. Indeed, θ is a scalar multiple of an algebra homomorphism in this case. As a corollary, zero-product preserving linear maps between standard operator algebras are proved to be of such form.

In Section 4, we deal with zero-product preserving linear maps between operator algebras. Since W^*-algebras are generated by idempotents, every such map between them is of the expected form (see also [15]). However, the C^*-algebra case is far more complicated. In [32], Wolff shows that if $\theta : A \to B$ is a bounded linear map between unital C^*-algebras preserving involution and zero products of self-adjoint elements in A then $\theta = \theta(1)J$ for a Jordan $*$-homomorphism J from A into B^{**}. We obtain a $*$-free version of this.

2. Zero-product preservers of algebras with dense subalgebra generated by idempotents. By the subalgebra of an algebra $\mathcal{M}$ generated by a subset S of $\mathcal{M}$ we mean the linear subspace of $\mathcal{M}$ spanned by the set of all finite products of elements in S.

LEMMA 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be topological algebras over a topological field $\mathbb{F}$ and $\theta : \mathcal{M} \to \mathcal{N}$ a continuous linear map preserving zero products. Suppose that the subalgebra of $\mathcal{M}$ generated by its idempotents is dense in $\mathcal{M}$. Then*

$$\theta(a)\theta(bc) = \theta(ab)\theta(c) \quad \textit{for all } a, b, c \in \mathcal{M}. \tag{2.1}$$

If, in addition, $\mathcal{M}$ *has an identity, then*

(i) $\theta(1)\theta(a) = \theta(a)\theta(1)$ *for all* $a \in \mathcal{M}$.

(ii) $\theta(1)\theta(ab) = \theta(a)\theta(b)$ *for all* $a, b \in \mathcal{M}$.

(iii) θ *preserves commutativity.*

Proof. Let $e, a, c \in \mathcal{M}$ with $e^2 = e$. As $ae(c - ec) = 0$, we have

$$0 = \theta(ae)\theta(c - ec) = \theta(ae)(\theta(c) - \theta(ec)),$$

and so

$$\theta(ae)\theta(c) = \theta(ae)\theta(ec). \tag{2.2}$$

On the other hand, from $(a - ae)ec = 0$, we obtain

$$0 = \theta(a - ae)\theta(ec) = (\theta(a) - \theta(ae))\theta(ec).$$

Therefore,

$$\theta(a)\theta(ec) = \theta(ae)\theta(ec). \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\theta(a)\theta(ec) = \theta(ae)\theta(c). \tag{2.4}$$

Moreover, for idempotents $e_1, \ldots, e_n$ in $\mathcal{M}$, we have

$$\theta(a)\theta(e_1 \ldots e_n c) = \theta(ae_1 \ldots e_n)\theta(c)$$

by applying (2.4) repeatedly. Since the subalgebra of $\mathcal{M}$ generated by its idempotents is dense and θ is a continuous map, we have

$$\theta(a)\theta(bc) = \theta(ab)\theta(c)$$

for all $a, b, c \in \mathcal{M}$. That is, (2.1) is established.

Suppose that $\mathcal{M}$ contains an identity. Then, by setting $a = 1$ in (2.1), we get

$$\theta(1)\theta(bc) = \theta(b)\theta(c)$$

for all $b, c \in \mathcal{M}$. That is, (ii) is established. And, setting $a = c = 1$ in (2.1), we have

$$\theta(1)\theta(b) = \theta(b)\theta(1)$$

for all $b \in \mathcal{M}$. That is, (i) is also established. Finally, by (ii) we have

$$\theta(1)\theta(ab - ba) = \theta(1)(\theta(ab) - \theta(ba)) = \theta(a)\theta(b) - \theta(b)\theta(a)$$

for all $a, b \in \mathcal{M}$. Then (iii) follows immediately. ∎

THEOREM 2.2. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be topological algebras over a topological field* $\mathbb{F}$ *and* $\theta : \mathcal{M} \to \mathcal{N}$ *a continuous linear map preserving zero products. Suppose that* $\mathcal{M}$ *has an identity and the subalgebra of* $\mathcal{M}$ *generated by idempotents is dense. Let* $\mathcal{N}'$ *be the subalgebra of* $\mathcal{N}$ *generated by* $\theta(\mathcal{M})$.

(i) *If* $\theta(1) = 0$ *then* $\theta(a)\theta(b) = 0$ *for all* $a, b \in \mathcal{M}$.

(ii) *If* $\theta(1)$ *is invertible in* $\mathcal{N}$, *or* $\mathcal{N}'$ *has an identity*, *then*

$$\theta(a) = h\varphi(a) \quad \textit{for all } a \in \mathcal{M},$$

where $h = \theta(1)$ *and* φ *is an algebra homomorphism from* $\mathcal{M}$ *into* $\mathcal{N}$.

Proof. (i) is a direct consequence of Lemma 2.1(ii).

(ii) Suppose first that $h = \theta(1)$ is invertible in $\mathcal{N}$. Let $\varphi = h^{-1}\theta$. Then it follows from Lemma 2.1 that φ is an algebra homomorphism from $\mathcal{M}$ into $\mathcal{N}$. Next, we assume that $\mathcal{N}'$ has an identity $1'$. Write

$$1' = \sum_{i=1}^{k} \theta(a_{i1})\theta(a_{i2})\dots\theta(a_{il_i})$$

for some a_{ij}'s in $\mathcal{M}$. Then

$$1' = 1'1' = \Big(\sum_{i=1}^{k} \theta(a_{i1})\theta(a_{i2})\dots\theta(a_{il_i})\Big)^2 = \theta(1)b = b\theta(1)$$

for some b in $\mathcal{N}'$ by Lemma 2.1. Thus $\theta(1)$ is invertible in $\mathcal{N}'$ and we are done. ∎

REMARK 2.3. The conclusion in Theorem 2.2(ii) need not be true in general even when $\theta : \mathcal{M} \to \mathcal{N}$ is bijective. For an example, consider $h_0 = (1, 1/2, 1/3, \dots, 1/n, \dots)$ in the abelian W^*-algebra $\mathcal{M} = \ell_\infty$. Let $\mathcal{N} = \{h_0 a : a \in \ell_\infty\}$. Then the bijective linear map $\theta(a) = h_0 a$ is continuous and zero-product preserving. However, $\mathcal{N}$ does not have an identity and $\theta(1)$ is not invertible. It is impossible to write $\theta = h\varphi$ for any central element h in $\mathcal{N}$ and for any algebra homomorphism φ from $\mathcal{M}$ into $\mathcal{N}$. However, the identity map would do the job if we are allowed to enlarge the co-domain of θ to ℓ_∞.

COROLLARY 2.4. *Let* $\mathbb{F}$ *be an algebraically closed field of characteristic* 0 *and* $\theta : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ *a linear map preserving zero products*, *where* n *and* r *are positive integers with* $n \geq 2$ *and* $n \geq r$. *Then either* $\operatorname{Im}(\theta)$, *the image of* θ, *has trivial multiplication*, *or* $n = r$ *and there exist an invertible matrix* $A \in M_n(\mathbb{F})$ *and a nonzero scalar* c *such that*

$$\theta(T) = cA^{-1}TA \quad \textit{for all } T \in M_n(\mathbb{F}).$$

Proof. Note that θ preserves commutativity by Lemma 2.1. We claim first that either $\operatorname{Im}(\theta)$ is commutative, or $n = r$ and $\theta(1)$ is in the center of $M_n(\mathbb{F})$. If $n \geq 3$, this follows immediately from [27, main theorem]. If $n = 2$, then either $r = 1$ in which case $\operatorname{Im}(\theta)$ is commutative, or $n = r = 2$ in which case the assertion follows from [27, Theorem 1.1].

Suppose first that $\operatorname{Im}(\theta)$ is commutative. Recall that

$$\theta(1)\theta(ab) = \theta(a)\theta(b) \quad \text{for all } a, b \in M_n(\mathbb{F})$$

from Lemma 2.1. We have

$$\theta(1)\theta(E_{kl}E_{ij}) = \theta(E_{kl})\theta(E_{ij}) = \theta(E_{ij})\theta(E_{kl}) = \theta(1)\theta(E_{ij}E_{kl}),$$

which is 0 if $i \neq l$ or $j \neq k$. Further, for any i, j, k, and l, we have

$$\begin{aligned}\theta(E_{ij})\theta(E_{ji}) &= \theta(1)\theta(E_{ii}) = \theta(E_{ik})\theta(E_{ki}) \\ &= \theta(E_{ki})\theta(E_{ik}) = \theta(1)\theta(E_{kk}) = \theta(E_{kl})\theta(E_{lk}).\end{aligned}$$

Therefore, $\mathrm{Im}(\theta)$ has nontrivial multiplication if and only if $\theta(E_{ij})\theta(E_{ji}) \neq 0$ for all i and j. Assume that $\mathrm{Im}(\theta)$ has nontrivial multiplication. For all k and l, we have

$$\theta(E_{kl})\sum_{i,j}\lambda_{ij}\theta(E_{ij}) = \sum_{i,j}\lambda_{ij}\theta(E_{kl})\theta(E_{ij}) = \lambda_{lk}\theta(1)\theta(E_{kk}).$$

Thus, the set $\{\theta(E_{ij}) \mid i,j = 1,\ldots,n\}$ is linearly independent, and the dimension of $\mathrm{Im}(\theta)$ is n^2. Since $r \leq n$ and $\mathrm{Im}(\theta)$ is commutative, this is not going to happen. Therefore, if $\mathrm{Im}(\theta)$ is commutative, then it has trivial multiplication.

Suppose next that $n = r$ and $c = \theta(1)$ is in the center of $M_n(\mathbb{F})$. If $c = 0$, then it follows from Lemma 2.1 again that $\mathrm{Im}(\theta)$ has trivial multiplication. Assume that $c \neq 0$. Then we see that $\varphi = c^{-1}\theta$ is a nonzero algebra homomorphism on $M_n(\mathbb{F})$. Note that $M_n(\mathbb{F})$ is a simple algebra over $\mathbb{F}$ and so φ is an $\mathbb{F}$-automorphism on $M_n(\mathbb{F})$. By the well-known Noether–Skolem theorem, $\varphi = c^{-1}\theta$ is an inner automorphism. This completes the proof. ∎

The following result, which is inspired by an example in [27, p. 310], shows that the condition $n \geq r$ is essential in the above result. Moreover, we see there is a third possibility for a zero-product preserving linear map other than being a product of a central element and an algebra homomorphism, or having the image with trivial multiplication.

Example 2.5. Let $\mathbb{F}$ be a field. Consider $\theta : M_k(\mathbb{F}) \to M_{k+2}(\mathbb{F})$ defined by

$$(a_{ij}) \mapsto \begin{pmatrix} 0 & a_{11} & a_{12} & \ldots & a_{1k} & 0 \\ 0 & 0 & 0 & \ldots & 0 & a_{11} \\ 0 & 0 & 0 & \ldots & 0 & a_{21} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{k1} \\ & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then θ is linear and preserves zero products. Since $\theta(E_{11})^2 \neq 0$, the image of θ carries a nontrivial multiplication. It is obvious that θ cannot be written as $c\varphi$ for any fixed element c in $M_{k+2}(\mathbb{F})$.

Although the main objects in this paper are continuous linear maps between topological algebras, results in this section can be strengthened to additive maps between rings in a few cases where the domain is a ring generated by its idempotents. For examples of such rings, see [29].

THEOREM 2.6. *Let $\mathcal{M}$ be a unital ring generated by its idempotents and θ a zero-product preserving additive map from $\mathcal{M}$ into a ring $\mathcal{N}$. Denote by $\mathcal{N}'$ the subring of $\mathcal{N}$ generated by $\theta(\mathcal{M})$. Then*

(i) $\theta(a)\theta(bc) = \theta(ab)\theta(c)$ *for all* $a, b, c \in \mathcal{M}$.
(ii) $\theta(1)\theta(a) = \theta(a)\theta(1)$ *for all* $a \in \mathcal{M}$.
(iii) $\theta(1)\theta(ab) = \theta(a)\theta(b)$ *for all* $a, b \in \mathcal{M}$.
(iv) θ *preserves commutativity.*
(v) *If* $\theta(1) = 0$ *then* $\theta(a)\theta(b) = 0$ *for all* $a, b \in \mathcal{M}$.
(vi) *If* $\theta(1)$ *is invertible in* $\mathcal{N}$, *or* $\mathcal{N}'$ *has an identity*, *then* $\theta(a) = \theta(1)\varphi(a)$ *for all* $a \in \mathcal{M}$, *where* φ *is a ring homomorphism from* $\mathcal{M}$ *into* $\mathcal{N}$.

Proof. By inspection of the proofs of Lemma 2.1 and Theorem 2.2, one can find that the continuity and linearity of θ are not used in this case. ∎

The following result of Pearcy and Topping [28] was brought to the authors' attention by Professor Pei-Yuan Wu (see also the survey paper of Wu [34]):

> *Every bounded linear operator on an infinite-dimensional complex Hilbert space H is a sum of at most five idempotents.*

Consequently, Theorem 2.6 applies to $\mathcal{M} = B(H)$, the algebra of all bounded linear operators on H, and thus provides a supplement to Corollary 2.4.

COROLLARY 2.7. *Let H be an infinite-dimensional complex Hilbert space and θ a zero-product preserving additive map from $B(H)$ into a ring $\mathcal{N}$. Let $\mathcal{N}'$ be the subring of $\mathcal{N}$ generated by $\theta(\mathcal{M})$. Then $\theta(1)$ is in the center of $\mathcal{N}'$ and*

$$\theta(1)\theta(ST) = \theta(S)\theta(T) \quad \text{for all } S, T \in B(H).$$

If, in addition, $\theta(1)$ is invertible in $\mathcal{N}$ or $\mathcal{N}'$ has an identity, then $\theta = \theta(1)\varphi$ where φ is a ring homomorphism from $B(H)$ into $\mathcal{N}$.

The following result should be known (and might be proved by other methods), although we do not have a handy reference.

COROLLARY 2.8. *Let H_1 and H_2 be infinite-dimensional complex Hilbert spaces and $\theta : B(H_1) \to B(H_2)$ a bijective additive map preserving zero products. Then*

$$\theta(T) = \lambda S^{-1}TS \quad \text{for all } T \in B(H_1),$$

where λ is a nonzero scalar and S is an invertible bounded linear operator from H_2 onto H_1.

Proof. By Corollary 2.7, $\theta(1)$ is in the center of $B(H_2)$ and so is a nonzero scalar λ. Then $\varphi = \lambda^{-1}\theta$ is a ring isomorphism from $B(H_1)$ onto $B(H_2)$. By [3, Theorem 4] there exists an invertible bounded linear operator S from H_2 onto H_1 such that $\varphi(T) = S^{-1}TS$ for all $T \in B(H_1)$. Thus the assertion follows. ∎

3. Zero-product preservers of algebras with matrix units. Let $\mathbb{F}$ be a topological field and $\mathcal{A}$ a topological algebra over $\mathbb{F}$. We call $\mathcal{A}$ a *prime algebra* if $x\mathcal{A}y = \{0\}$ implies $x = 0$ or $y = 0$. Let I be an index set, which can be of arbitrary cardinality. Suppose that E_{ij} exists in $\mathcal{A}$ for all i, j in I and satisfies the following conditions.

1. $E_{ij}E_{rs} = \delta_{jr}E_{is}$ for all i, j, r, s in I, where $\delta_{jr} = 1$ if $j = r$ and $\delta_{jr} = 0$ otherwise.
2. $\{E_{ij} : i, j \in I\}$ is *total* in $\mathcal{A}$, that is, the linear span of all E_{ij}'s is dense in $\mathcal{A}$.

In this case, we call $\{E_{ij} : i, j \in I\}$ a *complete system of matrix units* in $\mathcal{A}$.

THEOREM 3.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be topological algebras over a complete topological field $\mathbb{F}$ and $\theta : \mathcal{M} \to \mathcal{N}$ a continuous zero-product preserving linear map. Suppose that $\mathcal{M}$ has a complete system of matrix units $\{E_{ij} : i, j \in I\}$, and $\theta(\mathcal{M})$ is dense in a prime subalgebra $\mathcal{N}_0$ of $\mathcal{N}$. Then there is a nonzero λ in $\mathbb{F}$ and an algebra homomorphism φ from $\mathcal{M}$ into $\mathcal{N}$ such that $\theta = \lambda\varphi$. In particular, $\theta(\mathcal{M})$ is an algebra with a complete system of matrix units $\{\varphi(E_{ij}) : i, j \in I\}$.*

Proof. Set $A_{ij} = \theta(E_{ij})$ for i, j in I. Since $E_{ij}E_{st} = 0$ for $j \neq s$, we have

$$A_{ij}A_{st} = 0 \quad \text{if } j \neq s. \tag{3.1}$$

Let $\mathcal{N}'$ be the linear span of all A_{ij}'s. Since $\{E_{ij} : i, j \in I\}$ is total in $\mathcal{M}$ and $\theta(\mathcal{M})$ is dense in $\mathcal{N}_0$, the continuity of θ yields the density of $\mathcal{N}'$ in $\mathcal{N}_0$. Then the primeness of $\mathcal{N}_0$ implies that

$$x\mathcal{N}'y \neq 0 \quad \text{for all nonzero } x, y \in \mathcal{N}_0. \tag{3.2}$$

CLAIM 1. *For any $s, t \in I$, if $A_{st} \neq 0$ then $A_{ts} \neq 0$, $A_{ss} \neq 0$ and $A_{tt} \neq 0$.*

Indeed, it follows from $\{0\} \neq A_{st}\mathcal{N}'A_{st} \subseteq \mathbb{F}A_{st}A_{ts}A_{st}$ that $A_{ts} \neq 0$. Similarly, $\{0\} \neq A_{st}\mathcal{N}'A_{ts} \subseteq \mathbb{F}A_{st}A_{tt}A_{ts}$ and $\{0\} \neq A_{ts}\mathcal{N}'A_{st} \subseteq \mathbb{F}A_{ts}A_{ss}A_{st}$ imply $A_{tt} \neq 0$ and $A_{ss} \neq 0$ respectively.

CLAIM 2. *For any $i, k \in I$, $A_{ii}A_{ik} = \lambda_{ik}A_{ik}$ for some nonzero $\lambda_{ik} \in \mathbb{F}$.*

Since $\mathcal{N}'$ is dense in $\mathcal{N}_0$, we can find a net $\{A_\alpha\}_\alpha$ in $\mathcal{N}'$ such that

$$A_{ii}A_{ik} = \lim_\alpha A_\alpha.$$

Write $A_\alpha = \sum_{p,q\in I} \lambda_{pq\alpha} A_{pq}$ where $\lambda_{pq\alpha} \in \mathbb{F}$ and for each α at most finitely many $\lambda_{pq\alpha}$'s are nonzero.

First we show that $A_{ii}A_{ik} \in \mathbb{F}A_{ik}$. It suffices to consider the case when $A_{ii}A_{ik} \neq 0$. As

$$A_{ii}A_{ik} = \lim_\alpha \sum_{p,q\in I} \lambda_{pq\alpha} A_{pq}, \tag{3.3}$$

we have

$$A_{ii}A_{ii}A_{ik}A_{kk} = \lim_\alpha \lambda_{ik\alpha} A_{ii}A_{ik}A_{kk}.$$

Since $\mathbb{F}$ is complete, the net $\{\lambda_{ik\alpha}\}_\alpha$ converges to some λ_{ik} in $\mathbb{F}$. Define

$$x = A_{ii}A_{ik} - \lambda_{ik}A_{ik} = \lim_\alpha \sum_{(p,q)\neq(i,k)} \lambda_{pq\alpha} A_{pq}.$$

Then

$$A_{jl}xA_{st} = A_{jl}(A_{ii}A_{ik} - \lambda_{ik}A_{ik})A_{st} = 0 \qquad \text{for all } j,l,s,t \in I \text{ with } (l,s) \neq (i,k)$$

and

$$A_{ji}xA_{kt} = A_{ji}\Big(\lim_\alpha \sum_{(p,q)\neq(i,k)} \lambda_{pq\alpha} A_{pq}\Big)A_{kt} = 0 \quad \text{ for all } j,t \in I.$$

Hence $\mathcal{N}'x\mathcal{N}' = \{0\}$, and so $x = 0$. That is,

$$A_{ii}A_{ik} = \lambda_{ik}A_{ik} \in \mathbb{F}A_{ik}.$$

If $A_{ii}A_{ik} \neq 0$, certainly $\lambda_{ik} \neq 0$. On the other hand, if $A_{ii}A_{ik} = 0$ then

$$A_{ii}\mathcal{N}'A_{ik} \subseteq \mathbb{F}A_{ii}A_{ii}A_{ik} = \{0\},$$

and so $A_{ik} = 0$ by (3.2) and Claim 1. In this case, we can choose any nonzero λ_{ik} in $\mathbb{F}$. Thus Claim 2 is verified.

In the next step, we observe that for any i, j, k in I with $i \neq j$, it follows from

$$(E_{ii} + E_{ij})(E_{ik} - E_{jk}) = 0$$

that

$$(A_{ii} + A_{ij})(A_{ik} - A_{jk}) = 0.$$

As a consequence of (3.1) and Claim 2, we have

$$A_{ij}A_{jk} = A_{ii}A_{ik} = \lambda_{ik}A_{ik} \tag{3.4}$$

for some nonzero λ_{ik} in $\mathbb{F}$. Hence if $A_{ii} = 0$ or $A_{ij} = 0$ with $j \neq i$, then $\lambda_{ik}A_{ik} = 0$ and so $A_{ik} = 0$ for any $k \in I$. Similarly, if $A_{jk} = 0$ then $A_{ik} = 0$ for any $i \in I$ with $i \neq j$. Consequently, if $A_{st} = 0$ for some s and t in I, then $A_{ij} = 0$ for all i and j in I. In this case, $\mathcal{N}' = 0$ and hence θ is a zero map, which is not true. Therefore,

$$A_{st} \neq 0 \quad \text{for all } s, t \in I. \tag{3.5}$$

By (3.4), we have

$$(A_{ij}A_{jk})A_{kl} = \lambda_{ik}A_{ik}A_{kl} = \lambda_{ik}\lambda_{il}A_{il},$$
$$A_{ij}(A_{jk}A_{kl}) = \lambda_{jl}A_{ij}A_{jl} = \lambda_{jl}\lambda_{il}A_{il},$$

for some nonzero $\lambda_{ik}, \lambda_{il}, \lambda_{jl}$ in $\mathbb{F}$. Since $A_{il} \neq 0$, we have

$$\lambda_{ik} = \lambda_{jl} \quad \text{for all } i, j, k, l \in I.$$

Denote by λ this common nonzero value.

Finally, let $\varphi = \lambda^{-1}\theta$. By (3.1) and (3.4), we get

$$\varphi(E_{ij})\varphi(E_{st}) = 0 \qquad \text{for all } i, j, s, t \text{ with } j \neq s,$$
$$\varphi(E_{ij})\varphi(E_{jk}) = \varphi(E_{ik}) \quad \text{for all } i, j, k.$$

In other words,

$$\varphi(E_{ij}E_{kl}) = \varphi(E_{ij})\varphi(E_{kl}) \quad \text{for all } i, j, k, l \in I.$$

Since $\{E_{ij} : i, j \in I\}$ is total in $\mathcal{M}$ and φ is continuous, φ is actually an algebra homomorphism. It is clear that $\theta(\mathcal{M}) = \varphi(\mathcal{M})$ is an algebra with a complete system of matrix units $\{\varphi(E_{ij}) : i, j \in I\}$. ∎

COROLLARY 3.2. *Let H_i be a real or complex Hilbert space of arbitrary dimension, and let $K(H_i)$ be the algebra of all compact operators on H_i for $i = 1, 2$. Then every surjective norm continuous zero-product preserving linear map θ from $K(H_1)$ onto $K(H_2)$ is of the form*

$$\theta(T) = \lambda S^{-1}TS \quad \textit{for all } T \in K(H_1),$$

where λ is a nonzero scalar and S is an invertible bounded linear operator from H_2 onto H_1.

Proof. Note that $K(H_1)$ has a complete system of matrix units $\{e_i \otimes e_j : i, j \in I\}$, where $\{e_i : i \in I\}$ is an orthonormal basis of the Hilbert space H_1 and $e_i \otimes e_j$ is the rank one operator $x \mapsto \langle x, e_j \rangle e_i$ for x in H_1. Since $K(H_2)$ is prime, it follows from Theorem 3.1 that θ is a surjective algebra homomorphism φ multiplied by a nonzero scalar λ. Note that every nonzero continuous algebra homomorphism out of $K(H_1)$ is injective. Thus by [13, Corollary 3.2] the continuous algebra isomorphism φ between the operator algebras $K(H_1)$ and $K(H_2)$ is of the form $T \mapsto S^{-1}TS$ for some invertible bounded linear operator S from H_2 onto H_1. This completes the proof. ∎

In a similar manner, we can show the following

COROLLARY 3.3. *Let E, F be Banach spaces such that E has a Schauder basis. Then every surjective norm continuous zero-product preserving linear* *map θ from the Banach algebra $K(E)$ of compact operators on E onto $K(F)$ is of the form*

$$\theta(T) = \lambda S^{-1}TS \quad \text{for all } T \in K(E),$$

where λ is a nonzero scalar and S is an invertible bounded linear operator from F onto E.

COROLLARY 3.4. *Let $\langle E, F\rangle$ be two locally convex spaces in duality such that E (resp. F) is separable in the weak $\sigma(E,F)$ (resp. $\sigma(F,E)$) topology. Suppose $\mathcal{M}$ is an algebra of linear operators on E which are $\sigma(E,F)$-$\sigma(E,F)$ continuous, contains all continuous finite rank operators, and is equipped with the weak operator topology. Then every continuous zero-product preserving linear map θ from $\mathcal{M}$ into a prime topological algebra $\mathcal{N}$ with dense image is of the form $\theta = \lambda\varphi$ for a nonzero scalar λ and an algebra homomorphism φ from $\mathcal{M}$ into $\mathcal{N}$.*

Proof. We note that for such a dual pair $\langle E, F\rangle$, there is a biorthogonal system $\langle\{e_n\},\{f_n\}\rangle$, called the *Markushevich basis* for $\langle E, F\rangle$, such that $\{e_n : n = 1,2,\ldots\}$ is $\sigma(E,F)$ total in E, $\{f_n : n = 1,2,\ldots\}$ is $\sigma(F,E)$ total in F, and $\langle e_i, f_j\rangle = \delta_{ij}$ for $i,j = 1,2,\ldots$ (see [19, p. 289], and [25, p. 43] for the Banach space version). Let $e_i \otimes f_j$ be the rank one operator defined by $x \mapsto \langle x, f_j\rangle e_i$ for $i,j = 1,2,\ldots$ Then $\{e_i \otimes f_j : i,j = 1,2,\ldots\}$ is a complete system of matrix units for $\mathcal{M}$, and Theorem 3.1 applies. ∎

Recall that a subalgebra $\mathcal{S}$ of the algebra $B(E)$ of all bounded linear operators on a Banach space E is said to be *standard* if $\mathcal{S}$ contains all continuous finite rank operators. Since the dual of a separable Banach space is norm separable (see [19, p. 157]), the following result is a consequence of Corollary 3.4. Here, it is not necessary to assume that $\mathcal{S}$ contains the identity or is closed in any topology.

COROLLARY 3.5. *Let $\mathcal{S}_i$ be a standard operator algebra on a separable Banach space E_i equipped with the weak operator topology for $i = 1,2$. Let θ be a continuous zero-product preserving linear map from $\mathcal{S}_1$ into $\mathcal{S}_2$ with dense image. Then $\theta = \lambda\varphi$ for a nonzero scalar λ and an algebra homomorphism φ from $\mathcal{S}_1$ into $\mathcal{S}_2$.*

Recently, Araujo and Jarosz [2] showed that every bijective linear operator between two unital standard operator algebras which preserves zero products in both directions is a scalar multiple of an algebra isomorphism. However, in the nonbijective case it becomes a very difficult task without assuming continuity. Even discontinuous algebra homomorphisms have complicated structure ([22, 30]). In [33], one can find a purely algebraic approach to zero-product preserving maps.

4. Zero-product preservers of operator algebras. In this section we shall discuss zero-product preserving linear maps between operator algebras (over $\mathbb{C}$). Since the linear span of projections is norm dense in a W^*-algebra which is unital, we can apply Lemma 2.1 and Theorem 2.2 to get

THEOREM 4.1. *Let $\theta : \mathcal{M} \to \mathcal{N}$ be a norm continuous zero-product preserving linear map between W^*-algebras. Then*

(i) $\theta(a)\theta(bc) = \theta(ab)\theta(c)$ *for all* a, b *in* $\mathcal{M}$.
(ii) $\theta(1)\theta(a) = \theta(a)\theta(1)$ *for all* a, b *in* $\mathcal{M}$.
(iii) $\theta(1)\theta(ab) = \theta(a)\theta(b)$ *for all* a, b *in* $\mathcal{M}$.
(iv) θ *preserves commutativity.*
(v) *If* $\theta(1) = 0$ *then* $\theta(a)\theta(b) = 0$ *for all* a, b *in* $\mathcal{M}$.
(vi) *If* $\theta(1)$ *is invertible in* $\mathcal{N}$, *or* θ *is surjective*, *then*

$$\theta(a) = \theta(1)\varphi(a) \quad \text{for all } a \in \mathcal{M}, \tag{4.1}$$

where φ *is an algebra homomorphism from* $\mathcal{M}$ *into* $\mathcal{N}$.

We remark that part of Theorem 4.1 was obtained by Cui and Hou [15] using a different technique.

Recall that a W^*-algebra is called *properly infinite* if it contains no nonzero finite central projection. Since every element in a properly infinite W^*-algebra is a sum of at most five idempotents [28], Theorem 2.6 yields the following

THEOREM 4.2. *Let $\mathcal{M}$ be a properly infinite W^*-algebra and θ a zero-product preserving additive map from $\mathcal{M}$ into a unital ring $\mathcal{N}$. Then the conclusions in Theorem* 4.1 *hold*, *except that φ is a ring homomorphism from $\mathcal{M}$ into $\mathcal{N}$.*

Now, we turn to the C^*-algebra case. Since the linear sums of projections are dense in a unital C^*-algebra of real rank zero [11], the conclusions in Theorem 4.1 also hold for such C^*-algebras. However, to work with general C^*-algebras requires more effort.

Recall that a linear map J between two algebras is said to be a *Jordan homomorphism* if $J(xy + yx) = J(x)J(y) + J(y)J(x)$ for all x, y. In case the underlying field has characteristic not 2, this condition is equivalent to $J(x^2) = (Jx)^2$ for all x in the domain. The following result was obtained essentially by Wolff [32]. For a different approach, see also the thesis of J. Schweizer, *Interplay between noncommutative topology and operators on C^*-algebras*, Eberhard-Karls-Universität, Tübingen, 1997.

THEOREM 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{A}$ unital and $\theta : \mathcal{A}_{\mathrm{sa}} \to \mathcal{B}_{\mathrm{sa}}$ a zero-product preserving bounded linear map. Then*

$$\theta(a) = \theta(1)J(a) \quad \text{for all } a \text{ in } \mathcal{A}_{\mathrm{sa}},$$

where J is a Jordan homomorphism from $\mathcal{A}_{\mathrm{sa}}$ into $\mathcal{B}_{\mathrm{sa}}^{**}$ and $\theta(1)$ commutes with all the elements in the images of θ and J.

Note that as its original form that appeared in [32], the Jordan homomorphism J in the above theorem can be extended in a canonical way to a Jordan $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}^{**}$. In case $\theta(1)$ is invertible, the range of J is contained in $\mathcal{B}$. By a careful study of the original proof of Wolff in [32], we obtain a $*$-free version of Theorem 4.3. A great part of our arguments presented below is modeled on [32].

In what follows, we assume that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras with $\mathcal{A}$ unital and $\theta : \mathcal{A} \to \mathcal{B}$ is a bounded linear map such that $\theta(a)\theta(b) = 0$ for $a, b \in \mathcal{A}_{\mathrm{sa}}$ with $ab = 0$.

LEMMA 4.4. *For any a in $\mathcal{A}$, we have*

(i) $\theta(1)\theta(a) = \theta(a)\theta(1)$.
(ii) $\theta(1)\theta(a^2) = \theta(a)^2$.

In particular, if $\theta(1) = 0$ then $\theta(a)^2 = 0$ for all a in $\mathcal{A}$.

Proof. First, we note that it suffices to check (i) and (ii) for self-adjoint elements a of $\mathcal{A}$. Identify the C^*-subalgebra of $\mathcal{A}$ generated by 1 and a with $C(X)$, where $X \subseteq [-\|a\|, \|a\|]$ is the spectrum of a. Denote again by θ the bidual map of θ from $C(X)^{**}$ into $\mathcal{B}^{**}$. For each positive integer n and each integer k, let

$$X_{n,k} = (k/n, (k+1)/n] \cap X.$$

Pick an arbitrary point $x_{n,k}$ from each nonempty $X_{n,k}$. Set $x_{n,k} = \infty$ to be the isolated point at infinity of $X_\infty = X \cup \{\infty\}$ if $X_{n,k} = \emptyset$. For any $f \in C(X)$, using the convention $f(\infty) = 0$, we have

$$f = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} f(x_{n,k}) 1_{X_{n,k}}, \tag{4.2}$$

where $1_{X_{n,k}}$ is the characteristic function of the Borel set $X_{n,k}$, and the limit of the finite sums converges uniformly on X. In particular, for every fixed positive integer n we have

$$1 = \sum_{k\in\mathbb{Z}} 1_{X_{n,k}}. \tag{4.3}$$

For two disjoint nonempty sets $X_{n,j}$ and $X_{n,k}$, we can find two sequences $\{f_m\}_m$ and $\{g_m\}_m$ in $C(X)$ such that $f_{m+p}g_m = 0$ for $m, p = 0, 1, \ldots$, and $f_m \to 1_{X_{n,j}}$ and $g_m \to 1_{X_{n,k}}$ pointwise on X. By the weak* continuity of θ, we see that

$$\theta(1_{X_{n,j}})\theta(g_m) = \lim_{p\to\infty} \theta(f_{m+p})\theta(g_m) = 0 \quad \text{for all } m = 1, 2, \ldots$$

Thus

$$\theta(1_{X_{n,j}})\theta(1_{X_{n,k}}) = \lim_{m\to\infty} \theta(1_{X_{n,j}})\theta(g_m) = 0.$$

Consequently, for each positive integer n and each integer j we have

$$(4.4)\quad \theta(1)\theta(1_{X_{n,j}}) = \sum_{k\in\mathbb{Z}} \theta(1_{X_{n,k}})\theta(1_{X_{n,j}}) = \theta(1_{X_{n,j}})^2 = \theta(1_{X_{n,j}})\theta(1).$$

It follows from (4.2) and (4.4) that $\theta(1)\theta(f) = \theta(f)\theta(1)$ and

$$\begin{aligned}\theta(f)^2 &= \lim_{n\to\infty} \Big(\sum_{k\in\mathbb{Z}} f(x_{n,k})\theta(1_{X_{n,k}})\Big)^2 = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} f(x_{n,k})^2\theta(1_{X_{n,k}})^2 \\ &= \lim_{n\to\infty} \theta(1)\sum_{k\in\mathbb{Z}} f(x_{n,k})^2\theta(1_{X_{n,k}}) = \theta(1)\theta(f^2),\end{aligned}$$

for all f in $C(X)$. In particular, $\theta(1)\theta(a) = \theta(a)\theta(1)$ and $\theta(a)^2 = \theta(1)\theta(a^2)$. ∎

Lemma 4.5. *Suppose that $\mathcal{B}$ is also unital and $\theta(1)$ is invertible in $\mathcal{B}$. Then $\theta = \theta(1)\varphi$ for a bounded unital Jordan homomorphism φ from $\mathcal{A}$ into $\mathcal{B}$.*

Proof. By Lemma 4.4(i), $\theta(1)^{-1}$ commutes with $\theta(a)$ for all $a \in \mathcal{A}$. Then $\varphi = \theta(1)^{-1}\theta$ defines a bounded linear map from $\mathcal{A}$ into $\mathcal{B}$ with $\varphi(1) = 1$. The assertion now follows directly from Lemma 4.4(ii). ∎

Theorem 4.6. *Let θ be a surjective bounded linear map from a unital C^*-algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$. Suppose that θ sends zero products in $\mathcal{A}_{\mathrm{sa}}$ to zero products in $\mathcal{B}$. Then $\mathcal{B}$ is unital, $\theta(1)$ is invertible in the center of $\mathcal{B}$, and there is a surjective bounded unital Jordan homomorphism J from $\mathcal{A}$ onto $\mathcal{B}$ such that*

$$\theta(a) = \theta(1)J(a) \quad \textit{for all } a \in \mathcal{A}.$$

Proof. Since $\theta(1)\theta(a^2) = \theta(a)^2$ for all a in $\mathcal{A}$ and every element in a C^*-algebra is an algebraic sum of square elements, we see that $\theta(1)\mathcal{B} = \mathcal{B}$. In particular, $\theta(1)e = \theta(1)$ for some e in $\mathcal{B}$. Hence,

$$\theta(a)^2 e = \theta(1)\theta(a^2)e = \theta(a^2)\theta(1)e = \theta(a^2)\theta(1) = \theta(a)^2 \quad \text{for all } a \in \mathcal{A}.$$

Thus $be = b$ for all $b \in \mathcal{B}$. Similarly, $eb = b$ for all $b \in \mathcal{B}$. This implies that $\mathcal{B}$ is unital with identity e and it follows from $\theta(1)\mathcal{B} = \mathcal{B}$ that $\theta(1)$ is invertible. The last assertion follows from Lemma 4.5. ∎

It seems to be impossible to get a general result without assuming $\theta(1)$ being invertible. The following theorem might be an optimal one as we shall see in Example 4.8 after it. See also Example 4.10.

Theorem 4.7. *Let θ be a bounded linear map from a unital C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$. Suppose θ sends zero products in $\mathcal{A}_{\mathrm{sa}}$ to zero products in $\mathcal{B}$. If $\theta(1)$ is normal then there is a sequence of bounded Jordan* *homomorphisms J_n from $\mathcal{A}$ into $\mathcal{B}^{**}$ such that $\theta(1)J_n(a)$ converges to $\theta(a)$ strongly for all a in $\mathcal{A}$.*

Proof. We may assume that the $*$-subalgebra of $\mathcal{B}$ generated by $\theta(\mathcal{A})$ is norm dense in $\mathcal{B}$. Let $h = \theta(1)$. By Lemma 4.4 and the Fuglede–Putnam Theorem, both h and h^* are central elements in $\mathcal{B}$. For each positive integer n, let p_n be the central projection in $\mathcal{B}^{**}$ given by the spectral decomposition of h^*h corresponding to the set $\{\zeta \in \sigma(h^*h) : |\zeta| \geq 1/n\}$. Set $hn = p_n h$. Then h_n is an invertible central element of $p_n\mathcal{B}^{**}$, and θ_n given by $\theta_n(a) = p_n\theta(a)$ preserves zero products of self-adjoint elements. By Lemma 4.5, $J_n = h_n^{-1}\theta_n$ is a Jordan homomorphism from $\mathcal{A}$ into $p_n\mathcal{B}^{**}$ with $J_n(1) = p_n$. Note that $hJ_n(a) = h_nJ_n(a) = \theta_n(a)$. Since p_n converges strongly to the identity of $\mathcal{B}^{**}$, we see that $\theta_n(a)$ converges to $\theta(a)$ strongly for all a in $\mathcal{A}$. ∎

EXAMPLE 4.8. Let $\mathcal{M}$ be the C^*-algebra $\bigoplus_c M_2$ of all convergent sequences of 2×2 complex matrices. For each positive integer n we define the map $\theta_n : M_2 \to M_2$ by

$$\theta_n(A) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix},$$

and define $\theta : \mathcal{M} \to \mathcal{M}$ by

$$\theta\Big(\bigoplus_n A_n\Big) = \bigoplus_n \frac{\theta_n(A_n)}{\|\theta_n\|}.$$

It is clear that $\|\theta_n\| \geq 2n$ for $n = 1, 2, \ldots$ and θ is a zero-product preserving linear map of norm one. Note that $\theta(1) = \bigoplus_n I/\|\theta_n\|$ is in the center of $\mathcal{M}$, where I is the 2×2 identity matrix. Moreover, θ is not surjective, and $\theta(1)$ is normal with support projection 1 but not invertible. If we set $J_n(\bigoplus_k A_k) = \theta_1(A_1) \oplus \ldots \oplus \theta_n(A_n) \oplus 0 \ldots$, then J_n is a bounded Jordan homomorphism from $\mathcal{M}$ into $\mathcal{M}$ and $\theta(1)J_n(a)$ converges to $\theta(a)$ strongly for all a in $\mathcal{M}$. But there is no way to write $\theta = \theta(1)J$ for any Jordan homomorphism J from $\mathcal{M}$ into its second dual $\mathcal{M}^{**} = \bigoplus_{\ell^\infty} M_2$.

COROLLARY 4.9. *Let θ be a bounded linear map from a unital C^*-algebra $\mathcal{A}$ into an abelian C^*-algebra $\mathcal{B}$. Suppose that $\theta(a)\theta(b) = 0$ for all $a, b \in \mathcal{A}_{\mathrm{sa}}$ with $ab = 0$. Then there is an open projection p_1 in $\mathcal{B}^{**}$ such that*

(i) $\theta(1) = \theta(1)p_1$.

(ii) $\theta(\mathcal{A})p_0 = 0$, *where* $p_0 = 1 - p_1$.

(iii) $\theta(x)p_1 = \theta(1)\varphi(x)$ *for all* $x \in \mathcal{A}$, *where* $\varphi : \mathcal{A} \to p_1\mathcal{B}^{**}$ *is an algebra homomorphism.*

Consequently, $\theta = \theta(1)\varphi$.

Proof. Identify $\mathcal{B}$ with $C_0(Y)$ for a locally compact Hausdorff space Y. For each y in Y, denote by δ_y the point evaluation at y. Then the bounded linear functional $\delta_y \circ \theta$ is zero-product preserving. By Theorem 4.6, every nonzero $\delta_y\circ\theta$ is a scalar multiple of a unital complex Jordan homomorphism, and thus a character φ_y of $\mathcal{A}$. Write

$$\theta(a)(y) = h(y)\varphi_y(a) \quad \text{for all } a \in \mathcal{A}.$$

Note that $h(y) = 0$ if $\delta_y \circ \theta = 0$. This gives the function $h = \theta(1) \in C_0(Y)$. Let

$$Y_0 = \{y \in Y : h(y) = 0\} \quad \text{and} \quad Y_1 = Y \setminus Y_0.$$

Clearly, Y_0 is closed and Y_1 is open. Let p_i be the projection in $\mathcal{B}^{**}$ induced by the characteristic function of Y_i for $i = 0, 1$. Define $\varphi : \mathcal{A} \to p_1\mathcal{B}^{**}$ by

$$\varphi(a)(y) = \varphi_y(a) \quad \text{for all } a \in \mathcal{A} \text{ and } y \in Y_1.$$

Then the conclusion follows. ∎

The following example tells us that it is necessary to work with $\mathcal{B}^{**}$ rather than $\mathcal{B}$ for the co-domain of φ in the last corollary. In [21, Example 9], one can see that it is also necessary to assume $\mathcal{A}$ is unital.

Example 4.10. Let $\theta : C[0,1] \to C[0,1]$ defined by

$$\theta(f)(x) = \begin{cases} e^{-1/x} f\left(\sin\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then θ is a zero-product preserving bounded linear map. In the notation of Corollary 4.9, $Y_0 = \{0\}$ and $Y_1 = (0, 1]$. Note that $h = \theta(1) \in C[0, 1]$. But the range of φ is not contained in $C[0, 1]$.

In case θ preserves all zero products in $\mathcal{A}$, we have the following

Theorem 4.11. *Let θ be a surjective bounded linear map from a unital C^*-algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$. Suppose that $\theta(a)\theta(b) = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. Then $\mathcal{B}$ is unital and $\theta(1)$ is an invertible element in the center of $\mathcal{B}$. Moreover, $\theta = \theta(1)\varphi$ for a surjective algebra homomorphism φ from $\mathcal{A}$ onto $\mathcal{B}$.*

To prove Theorem 4.11, we need the following results.

First, recall that an algebra $\mathcal{A}$ is *semiprime* if $x\mathcal{A}x \neq 0$ for all nonzero $x \in \mathcal{A}$. An ideal I of an algebra $\mathcal{A}$ is said to be *essential* if $I \cap J \neq 0$ for every nonzero ideal J of $\mathcal{A}$. For semiprime algebras, this is equivalent to the condition that $Ia = 0$ implies $a = 0$ for all a in $\mathcal{A}$. For a Jordan homomorphism θ, we denote by $\ker\theta$ the kernel of θ, i.e., $\ker\theta = \{x : \theta(x) = 0\}$.

Theorem 4.12 ([5, Theorem 2.3]). *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over any field of characteristic not 2 with $\mathcal{B}$ semiprime and θ a Jordan homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then there exist ideals U and V of $\mathcal{A}$ and ideals U' and V' of $\mathcal{B}$ such that*

(i) $U \cap V = \ker\theta$ *and* $U + V$ *is an essential ideal of* $\mathcal{A}$.
(ii) $U' \cap V' = 0$ *and* $U' \oplus V'$ *is an essential ideal of* $\mathcal{B}$.
(iii) $\theta(U) = U'$ *and* $\theta(V) = V'$.
(iv) $\theta(ux) = \theta(u)\theta(x)$ *for all* $u \in U$, $x \in \mathcal{A}$.
(v) $\theta(vx) = \theta(x)\theta(v)$ *for all* $v \in V$, $x \in \mathcal{A}$.

Moreover, if $\mathcal{B}$ *is a normed algebra, then the ideals* U' *and* V' *are closed.*

In general we cannot claim that $U \oplus V = \mathcal{A}$ and $U' \oplus V' = \mathcal{B}$ (see the example in [4, p. 458] which the authors attribute to Kaplansky).

THEOREM 4.13 (Brešar). *Let* θ *be a Jordan isomorphism from a (complex) algebra* $\mathcal{A}$ *onto a* C^**-algebra* $\mathcal{B}$. *If* θ *preserves zero products, then* θ *is an isomorphism.*

Proof. Let U, V, U', V' be as in Theorem 4.12. Since V' is a closed ideal of $\mathcal{B}$, V' itself is a C^*-algebra. We claim that V' is commutative. If not, then by Kaplansky's theorem [23, p. 292] there would be $a \in V'$ such that $a \neq 0$ and $a^2 = 0$. Set $b = \theta^{-1}(a) \in V$ and $c = \theta^{-1}(aa^*) \in V$. By Theorem 4.12(v) we have $\theta(cb) = \theta(b)\theta(c) = a^2a^* = 0$ and so $cb = 0$. But then $aa^*a = \theta(c)\theta(b) = 0$, a contradiction. Thus V' is indeed commutative.

We claim that V' lies in the center of $\mathcal{B}$. Given $a, b \in V'$ and $x \in \mathcal{B}$ we see that a commutes with b and with bx, so that $0 = [a, bx] = b[a, x]$, that is, $V'[\mathcal{B}, V'] = 0$. Here $[a, b] = ab - ba$ is the Lie product of a and b. In particular, $[\mathcal{B}, V']\mathcal{B}[\mathcal{B}, V'] = 0$, which implies that $[\mathcal{B}, V'] = 0$, proving our claim. Accordingly, Theorem 4.12(iv) and (v) together show that $\theta(wx) = \theta(w)\theta(x)$ for all $x \in \mathcal{A}$ and all w from the essential ideal $W = U \oplus V$. Consequently, for all $w \in W$ and $x, y \in \mathcal{A}$ we have

$$\theta(w)\theta(xy) = \theta(wxy) = \theta((wx)y) = \theta(wx)\theta(y) = \theta(w)\theta(x)\theta(y),$$

that is, $\theta(W)(\theta(xy) - \theta(x)\theta(y)) = 0$ for all $x, y \in \mathcal{A}$. Since $\theta(W) = U' \oplus V'$ is an essential ideal of $\mathcal{B}$ by Theorem 4.12(ii), it follows that $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in \mathcal{A}$, that is, θ is an isomorphism. ∎

Theorem 4.13 is an unpublished result due to Professor Matej Brešar, who kindly agreed to it being included here. On the other hand, the proof of the following elementary result of C. A. Akemann and G. K. Pedersen was communicated by Professor Lawrence G. Brown.

LEMMA 4.14 (Akemann and Pedersen). *Let* I *be a closed two-sided ideal of a* C^**-algebra* $\mathcal{A}$. *Let* $\overline{x} = x + I \in \mathcal{A}/I$ *and* $\overline{y} = y + I \in \mathcal{A}/I$ *be such that* $\overline{xy} = 0$. *Then there are* $a, b \in \mathcal{A}$ *such that* $\overline{x} = a + I$, $\overline{y} = b + I$ *and* $ab = 0$.

Proof. Observe that $\overline{xy} = 0$ exactly when $|\overline{x}|\,|\overline{y}^*| = 0$, where $|\overline{x}| = (\overline{x}^*\overline{x})^{1/2}$ is the absolute value of $\overline{x}$. In this case, $\overline{z} = |\overline{x}| - |\overline{y}^*|$ is self-adjoint. As $z + I = \frac{1}{2}(z + z^*) + I$, we may assume that z is a self-adjoint element in $\mathcal{A}$. Write $z = c - d$ as the difference of its positive and negative parts.

Note $cd = dc = 0$. It is easy to see that $|\overline{x}| = c + I$ and $|\overline{y}^*| = d + I$. Let $0 < \alpha < 1$. By a standard result (see, e.g., [24, Proposition 2.9.2]), there exist $\overline{u}, \overline{v}$ in $\mathcal{A}/I$ such that $\overline{x} = \overline{u}\,|\overline{x}|^\alpha$ and $\overline{y}^* = \overline{v}\,|\overline{y}^*|^\alpha$. Then $a = uc^\alpha$ and $b^* = vd^\alpha$ will do the job. ∎

Proof of Theorem 4.11. Taking into account Theorem 4.6, we only need to show that φ is a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Let I be the kernel of φ, which is a closed Jordan ideal of the C^*-algebra $\mathcal{A}$. By [14], I is also a two-sided ideal of $\mathcal{A}$. Then φ induces a Jordan isomorphism from the quotient algebra $\mathcal{A}/I$ onto B, which is again zero-product preserving by Lemma 4.14. By Theorem 4.13, this is an algebra isomorphism. Consequently, φ is a surjective algebra homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. ∎

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