Superimposed optimization methods for the mixed equilibrium problem and variational inclusion

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Abstract

The purpose of this paper is to construct two superimposed optimization methods for solving the mixed equilibrium problem and variational inclusion. We show that the proposed superimposed methods converge strongly to a solution of some optimization problem. Note that our methods do not involve any projection.

Keywords: Mixed equilibrium problem; Variational inclusion; Monotone operator; Superimposed optimization method; Resolvent.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping and $K : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find an x in C such that

$$K(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C.$$

The theory of equilibrium problems provides us a natural, novel and unified framework to study a wide class of applications. The ideas and techniques involved are being used in a variety of diverse areas and proved to be productive and innovative. It has been shown by Blum and Oettli [1] and Noor and Oettli [15] that variational inequalities and mathematical programming problems can be viewed as special realizations of

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abstract equilibrium problems. Equilibrium problems have numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics. There are a great number of numerical methods for solving equilibrium problems under various assumptions on K and B. Please see, e.g., [2, 4, 5, 7, 11, 12, 13, 17, 18, 21, 24, 27, 28, 29, 31, 32, 33, 34].

In 1997, Combettes and Hirstoaga [6] introduced an iterative method solving equilibrium problems, and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [23], Yao, Liou and Yao [30], and Zeng and Yao [35] considered several iterative schemes for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of a family of finitely or infinite many nonexpansive mappings. Moreover, Marino, Cianciaruso, Muglia and Yao [3] presented an iterative method for finding common solutions of an equilibrium problem and a variational inequality.

On the other hand, let $A : H \to H$ be a single-valued nonlinear mapping and $R : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion, which is to find a point x in H such that

$$0 \in A(x) + R(x), \tag{1.1}$$

where 0 is the zero vector in H. The set of solutions of problem (1.1) is denoted by $(A + R)^{-1}0$. If $H = \mathbb{R}^m$, problem (1.1) becomes the generalized equation introduced by Robinson [19]. If $A = 0$, problem (1.1) becomes the inclusion problem introduced by Rockafellar [20]. Problem (1.1) provides a convenient framework for a unified study of mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, etc. Meanwhile, various types of variational inclusion problems have been extended and generalized.

In this paper, we are interested in solving the equilibrium problem with those K given by

$$K(x, y) = F(x, y) + G(x, y),$$

where $F, G : C \times C \to \mathbb{R}$ are two bifunctions satisfying some special properties (see Section 2). This is the well-known mixed equilibrium problem, i.e., to find an x in C such that

$$F(x, y) + G(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by $EP(F, G, B)$.

Recently, Zhang et al. [36] introduced a new iterative scheme for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [16] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping. For more related works, please see also, [8, 9, 10].

Motivated and inspired by the above works, the purpose of this paper is to construct two superimposed optimization methods for solving the mixed equilibrium problem and variational inclusion. Consequently, we show that the suggested superimposed methods converge strongly to a solution of some optimization problem. Note that our methods do not use any projection.

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2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We write $x_n \rightharpoonup x$ and $x_n \to x$ for the weak and strong convergence of $\{x_n\}$ to x in H, respectively. The set of fixed points of a mapping T on a subset C of H is $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Let R be a mapping of H into 2^H. The *effective domain* of R is

$$\mathrm{dom}(R) = \{x \in H : Rx \neq \emptyset\}.$$

A multi-valued mapping R is said to be a *monotone* operator on H if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall x, y \in \mathrm{dom}(R), \forall u \in Rx, \forall v \in Ry.$$

A monotone operator R on H is said to be *maximal* if its graph is not strictly contained in the graph of any other monotone operator on H. Let R be a maximal monotone operator on H and let $R^{-1}0 = \{x \in H : 0 \in Rx\}$.

For a maximal monotone operator R on H and $\lambda > 0$, we may define a single-valued operator

$$J_\lambda^R = (I + \lambda R)^{-1} : H \to \mathrm{dom}(R),$$

which is called the *resolvent* of R for λ. It is known that the resolvent J_λ^R is *firmly nonexpansive*, i.e.,

$$\|J_\lambda^R x - J_\lambda^R y\|^2 \leq \langle J_\lambda^R x - J_\lambda^R y, x - y \rangle, \quad \forall x, y \in C,$$

and $R^{-1}0 = \mathrm{Fix}(J_\lambda^R)$ for all $\lambda > 0$.

Let C be a nonempty closed convex subset of real Hilbert space H. Recall that a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is *α-inverse-strongly monotone* if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Throughout this paper, we assume that two bifunctions $F, G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(F1) $F(x,x)=0$ for all x in C;

(F2) F is *monotone*, i.e., $F(x,y)+F(y,x)\le 0$ for all $x,y\in C$;

(F3) for each x,y,z in C, we have $\limsup_{t\to 0^+} F(tz+(1-t)x,y)\le F(x,y)$;

(F4) for each x in C, the function $y\mapsto F(x,y)$ is convex and weakly lower semicontinuous.

(G1) $G(x,x)=0$ for all x in C;

(G2) G is monotone, and weakly upper semicontinuous in the first variable;

(G3) G is convex in the second variable.

(H) For fixed $\mu>0$ and x in C, there exists a bounded set $K\subset C$ and a in K such that

$$-F(a,z)+G(z,a)+\frac{1}{\mu}\langle a-z,z-x\rangle<0,\quad \forall z\in C\backslash K.$$

Lemma 2.1. *([3]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F,G:C\times C\to\mathbb{R}$ be two bifunctions which satisfy conditions (F1)-(F4), (G1)-(G3) and (H). Let $\mu>0$ and $x\in C$. Then, there exists a unique $z:=T_\mu x$ in C such that*

$$F(z,y)+G(z,y)+\frac{1}{\mu}\langle y-z,z-x\rangle\ge 0,\quad \forall y\in C.$$

Furthermore,

(i) The single-valued map T_μ thus defined is firmly nonexpansive, i.e., for any x,y in H, we have

$$\|T_\mu x-T_\mu y\|^2\le\langle T_\mu x-T_\mu y,x-y\rangle,\quad \forall x,y\in H.$$

(ii) $EP(F,G,0)$ is closed and convex, and $EP(F,G,0)=\mathrm{Fix}(T_\mu)$.

Lemma 2.2. ([14]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let a mapping $A:C\to H$ be α-inverse strongly monotone and $\lambda>0$ be a constant. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2\le\|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2,\quad \forall x,y\in C.$$

In particular, if $0\le\lambda\le 2\alpha$, then $I-\lambda A$ is nonexpansive.

Lemma 2.3. *([26]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\le(1-\gamma_n)a_n+\delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n=+\infty$;

(2) $\limsup_{n\to\infty}\delta_n/\gamma_n\le 0$ or $\sum_{n=1}^{\infty}|\delta_n|<+\infty$.

Then $\lim_{n\to\infty}a_n=0$.

3 Results

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A, B be two nonlinear operators of C into H. Let R be a maximal monotone operator on H with the resolvent $J_\lambda^R = (I + \lambda R)^{-1}$. Let $F, G : C \times C \to \mathbb{R}$ be two bifunctions. Set

$$\Gamma := EP(F, G, B) \cap (A + R)^{-1}0.$$

We consider the following optimization problem:

$$\min\{\|x\| : x \in \Gamma\}, \tag{3.1}$$

the least-squares solutions to the constrained linear inverse problem.

Throughout, we assume $\Gamma \neq \emptyset$ together with the following conditions.

(1) $0 < a \leq \lambda \leq b < 2\alpha$ and $0 < c \leq \mu \leq d < 2\beta$ are all constants.

(2) A, B are α, β-inverse strongly-monotone mappings of C into H, respectively.

(3) The effective domain of R is included in C.

(4) F and G satisfy conditions (F1)-(F4), (G1)-(G3) and (H).

Let T_μ be the single-valued firmly nonexpansive map defined in Lemma 2.1. It is easy to see that

$$z = T_\mu(I - \mu B)z = J_\lambda^R(I - \lambda A)z, \quad \forall z \in \Gamma. \tag{3.2}$$

In order to solve (3.1), we introduce the following superimposed optimization method:

$$x_t = T_\mu(I - \mu B)J_\lambda^R((1 - t)I - \lambda A)x_t, \quad t \in (0, 1 - \frac{\lambda}{2\alpha}).$$

Note that the net $\{x_t\}$ above is well-defined. More precisely, for any t in $(0, 1 - \frac{\lambda}{2\alpha})$, we define a mapping

$$S_t := T_\mu(I - \mu B)J_\lambda^R((1 - t)I - \lambda A).$$

Since T_μ, J_λ^R, $I - \mu B$ and $I - \frac{\lambda}{1-t}A$ (see Lemmas 2.2 and 2.2) are all nonexpansive, we have, for any x, y in C,

$$\begin{aligned}
\|S_t x - S_t y\| &= \left\|T_\mu(I - \mu B)J_\lambda^R((1 - t)I - \lambda A)x - T_\mu(I - \mu B)J_\lambda^R((1 - t)I - \lambda A)y\right\| \\
&\leq \|((1 - t)I - \lambda A)x - ((1 - t)I - \lambda A)y\| \\
&= (1 - t)\|(x - \frac{\lambda}{1 - t}Ax) - (y - \frac{\lambda}{1 - t}Ay)\| \\
&\leq (1 - t)\|x - y\|.
\end{aligned}$$

Thus, the mapping S_t is a contraction on C, and consequently there exists a unique fixed point x_t of S_t in C.

Theorem 3.1. *The net $\{x_t\}$ defined by*

$$x_t = T_\mu(I-\mu B)J_\lambda^R((1-t)I-\lambda A)x_t, \quad t\in(0,1-\frac{\lambda}{2\alpha}). \tag{3.3}$$

converges strongly, as $t\to 0^+$, to the unique point x^* in Γ of minimum norm. Moreover, we have

$$\|x^*\|^2 \le \langle x^*, z\rangle, \quad \forall z\in\Gamma.$$

Proof. **Step 1.** We first show that $\{x_t\}$ and $\{Ax_t\}$ are both uniformly bounded, and

$$\lim_{t\to 0^+}\|Ax_t - Az\| = 0, \quad \forall z\in\Gamma. \tag{3.4}$$

Let $z\in\Gamma$. Note that

$$z = J_\lambda^R(z-\lambda Az) = J_\lambda^R\Big(tz + (1-t)(z-\lambda Az/(1-t))\Big).$$

From (3.2), (3.3), the nonexpansivity of T_μ, J_λ^R, $I-\mu B$ and $I-\lambda A/(1-t)$ (Lemmas 2.1 and 2.2), and the convexity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}
&\|x_t - z\|^2\\
=\;& \|T_\mu(I-\mu B)J_\lambda^R((1-t)I-\lambda A)x_t - T_\mu(I-\mu B)z\|^2\\
\le\;& \Big\|J_\lambda^R\Big((1-t)x_t-\lambda Ax_t\Big) - z\Big\|^2 && (3.5)\\
=\;& \Big\|J_\lambda^R\Big((1-t)(x_t-\lambda Ax_t/(1-t))\Big) - J_\lambda^R\Big(tz+(1-t)(z-\lambda Az/(1-t))\Big)\Big\|^2\\
\le\;& \Big\|\Big((1-t)(x_t-\lambda Ax_t/(1-t))\Big) - \Big(tz+(1-t)(z-\lambda Az/(1-t))\Big)\Big\|^2\\
=\;& \Big\|(1-t)\Big((x_t-\lambda Ax_t/(1-t)) - (z-\lambda Az/(1-t))\Big) + t(-z)\Big\|^2 && (3.6)\\
\le\;& (1-t)\|(x_t-\lambda Ax_t/(1-t)) - (z-\lambda Az/(1-t))\|^2 + t\|z\|^2 && (3.7)\\
\le\;& (1-t)\|x_t - z\|^2 + t\|z\|^2
\end{aligned}$$

It follows that

$$\|x_t - z\| \le \|z\|.$$

Therefore, $\{x_t\}$ is bounded. Since A is α-inverse strongly monotone, it is $\frac{1}{\alpha}$-Lipschitz continuous. We deduce immediately that $\{Ax_t\}$ is also bounded.

On the other hand, from the α-inverse strong monotonicity of A, we derive

$$\begin{aligned}
&\|(x_t-\lambda Ax_t/(1-t)) - (z-\lambda Az/(1-t))\|^2\\
=\;& \|(x_t - z) - \lambda(Ax_t - Az)/(1-t)\|^2\\
=\;& \|x_t - z\|^2 - \frac{2\lambda}{1-t}\langle Ax_t - Az, x_t - z\rangle + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2\\
\le\;& \|x_t - z\|^2 - \frac{2\alpha\lambda}{1-t}\|Ax_t - Az\|^2 + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2\\
=\;& \|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2. && (3.8)
\end{aligned}$$

It follows from (3.7) and (3.8) that

$$\|x_t - z\|^2 \leq (1-t)\Big(\|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2\Big) + t\|z\|^2.$$

Consequently,

$$\frac{\lambda}{(1-t)}(2(1-t)\alpha - \lambda)\|Ax_t - Az\|^2 \leq t\|z\|^2 \to 0, \quad \text{as } t \to 0^+.$$

As a result,

$$\lim_{t\to 0^+} \|Ax_t - Az\| = 0.$$

Step 2. Next, we show

$$\lim_{t\to 0^+} \Big\|x_t - J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big)\Big\| = 0. \tag{3.9}$$

Using the firm nonexpansivity of J_λ^R, for any z in Γ we have

$$\begin{aligned}
&\Big\|J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\
= \;& \Big\|J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - J_\lambda^R\Big(z - \lambda Az\Big)\Big\|^2 \\
\leq \;& \Big\langle (1-t)x_t - \lambda Ax_t - (z - \lambda Az), J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\rangle \\
= \;& \frac{1}{2}\Big(\|(1-t)x_t - \lambda Ax_t - (z - \lambda Az)\|^2 + \Big\|J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\
& -\Big\|(1-t)x_t - \lambda(Ax_t - \lambda Az) - J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big)\Big\|^2\Big).
\end{aligned}$$

By the nonexpansivity of $I - \lambda A/(1-t)$, we have

$$\begin{aligned}
&\|(1-t)x_t - \lambda Ax_t - (z - \lambda Az)\|^2 \\
= \;& \|(1-t)((x_t - \lambda Ax_t/(1-t) - (z - \lambda Az/(1-t))) + t(-z)\|^2 \\
\leq \;& (1-t)\|(x_t - \lambda Ax_t/(1-t) - (z - \lambda Az/(1-t))\|^2 + t\|z\|^2 \\
\leq \;& (1-t)\|x_t - z\|^2 + t\|z\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big\|J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\
\leq \;& \frac{1}{2}\Big((1-t)\|x_t - z\|^2 + t\|z\|^2 + \Big\|J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - z\Big\|^2 \\
& -\Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda Ax_t\Big) - \lambda(Ax_t - Az)\Big\|^2\Big).
\end{aligned}$$

This together with (3.5) gives

$$\begin{aligned}
&\|x_t - z\|^2 \\
\leq\ & \Big\|J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big) - z\Big\|^2 \\
\leq\ & (1-t)\|x_t - z\|^2 + t\|z\|^2 - \Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big) - \lambda(Ax_t - Az)\Big\|^2 \\
=\ & (1-t)\|x_t - z\|^2 + t\|z\|^2 - \Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\|^2 \\
& +2\lambda\Big\langle(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big), Ax_t - Az\Big\rangle - \lambda^2\|Ax_t - Az\|^2 \\
\leq\ & (1-t)\|x_t - z\|^2 + t\|z\|^2 - \Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\|^2 \\
& +2\lambda\Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\|\|Ax_t - Az\|. \qquad (3.10)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\|^2 \\
\leq\ & t\|z\|^2 + 2\lambda\Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\|\|Ax_t - Az\|.
\end{aligned}$$

Since $\|Ax_t - Az\| \to 0$ as $t \to 0^+$ by Step 1, we deduce

$$\lim_{t\to 0^+}\Big\|(1-t)x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\| = 0.$$

Therefore,

$$\lim_{t\to 0^+}\Big\|x_t - J_\lambda^R\Big((1-t)x_t - \lambda A x_t\Big)\Big\| = 0.$$

Step 3. We show that $\|x_t - T_\mu(I-\mu B)x_t\| \to 0$ as $t \to 0^+$.

Set

$$u_t = J_\lambda^R((1-t)x_t - \lambda A x_t), \quad \forall t \in (0, 1 - \frac{\lambda}{2\alpha}).$$

Then by (3.3),

$$x_t = T_\mu(I-\mu B)u_t.$$

It follows from Step 2 and the nonexpansivity of T_μ and $I - \mu B$ that

$$\begin{aligned}
\|x_t - T_\mu(I-\mu B)x_t\| &= \|T_\mu(I-\mu B)u_t - T_\mu(I-\mu B)x_t\| \\
&\leq \|u_t - x_t\| \to 0.
\end{aligned}$$

Step 4. We show that $\{x_t\}$ strongly converges to the unique point x^* in Γ of minimum norm as $t \to 0^+$.

From (3.6) and the nonexpansivity of $I - \frac{\lambda}{1-t}A$, we have

$$\begin{aligned}
\|x_t - z\|^2 &\leq \Big\|(1-t)\Big((x_t - \frac{\lambda}{1-t}Ax_t) - (z - \frac{\lambda}{1-t}Az)\Big) - tz\Big\|^2 \\
&= (1-t)^2\Big\|(x_t - \frac{\lambda}{1-t}Ax_t) - (z - \frac{\lambda}{1-t}Az)\Big\|^2 \\
&\quad -2t(1-t)\Big\langle z, (x_t - \frac{\lambda}{1-t}Ax_t) - (z - \frac{\lambda}{1-t}Az)\Big\rangle + t^2\|z\|^2 \\
&\leq (1-t)^2\|x_t - z\|^2 - 2t(1-t)\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + t^2\|z\|^2 \\
&= (1-2t)\|x_t - z\|^2 + 2t\Big\{ - (1-t)\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle\Big\} \\
&\quad +t^2(\|z\|^2 + \|x_t - z\|^2).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_t - z\|^2 &\leq -\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + \frac{t}{2}(\|z\|^2 + \|x_t - z\|^2) \\
&\quad +t\|z\|\Big\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\| \\
&\leq -\Big\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\rangle + tM. \qquad (3.11)
\end{aligned}$$

Here, M is some constant such that

$$\frac{\|z\|^2 + \|x_t - z\|^2}{2} + \|z\|\Big\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\Big\| \leq M, \quad \forall t \in (0, 1 - \frac{\lambda}{2\alpha}).$$

Let x^* be any weak* cluster point in C of the bounded net $\{x_t\}$ as $t \to 0^+$. Assume $t_n \to 0^+$ in $(0, 1 - \frac{\lambda}{2\alpha})$ as $n \to \infty$, and $x_{t_n} \rightharpoonup x^*$. Put $x_n := x_{t_n}$ and $u_n := u_{t_n}$. From (3.11), we have

$$\|x_n - z\|^2 \leq -\Big\langle z, x_n - \frac{\lambda}{1-t_n}(Ax_n - Az) - z\Big\rangle + t_nM, \quad \forall z \in \Gamma. \qquad (3.12)$$

Now we show $x^* \in EP(F, G, B)$. Setting

$$v_n = T_\mu(I - \mu B)x_n,$$

for any y in C we have

$$F(v_n, y) + G(v_n, y) + \frac{1}{\mu}\langle y - v_n, v_n - (x_n - \mu Bx_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$G(v_n, y) + \langle y - v_n, \frac{v_n - x_n}{\mu} + Bx_n\rangle \geq F(y, v_n), \quad \forall y \in C. \qquad (3.13)$$

Put $z_s = sy + (1-s)x^* \in C$ for s in $(0,1]$ and y in C. It follows from the β-inverse strongly monotonicity of B and (3.13) that

$$\begin{aligned}
& \langle z_s - v_n, Bz_s\rangle \\
= \;& \langle z_s - v_n, Bz_s - Bx_n\rangle + \langle Bx_n, z_s - v_n\rangle \\
\geq \;& \langle z_s - v_n, Bz_s - Bx_n\rangle + F(z_s, v_n) - G(v_n, z_s) - \frac{1}{\mu}\langle z_s - v_n, v_n - x_n\rangle \\
= \;& \langle z_s - v_n, Bz_s - Bv_n\rangle + \langle z_s - v_n, Bv_n - Bx_n\rangle \\
& +F(z_s, v_n) - G(v_n, z_s) - \frac{1}{\mu}\langle z_s - v_n, v_n - x_n\rangle \\
\geq \;& \langle z_s - v_n, Bv_n - Bx_n\rangle + F(z_s, v_n) - G(v_n, z_s) - \frac{1}{\mu}\langle z_s - v_n, v_n - x_n\rangle. \quad (3.14)
\end{aligned}$$

Note that $\|Bv_n - Bx_n\| \leq \frac{1}{\beta}\|v_n - x_n\| \to 0$ by Step 3, and also that $v_n \rightharpoonup x^*$ weakly. Letting $n \to \infty$ in (3.14), we have from the assumptions (F4) and (G2) that

$$\langle z_s - x^*, Bz_s\rangle \geq F(z_s, x^*) - G(x^*, z_s). \quad (3.15)$$

From (F1), (F3), (G1), (G2), (G3) and (3.15), we also have

$$\begin{aligned}
0 &= F(z_s, z_s) + G(z_s, z_s) \\
&\leq sF(z_s, y) + (1-s)F(z_s, x^*) + sG(z_s, y) + (1-s)G(z_s, x^*) \\
&\leq sF(z_s, y) + sG(z_s, y) + (1-s)[F(z_s, x^*) - G(x^*, z_s)] \\
&\leq sF(z_s, y) + sG(z_s, y) + (1-s)\langle z_s - x^*, Bz_s\rangle \\
&= s[F(z_s, y) + G(z_s, y) + (1-s)\langle y - x^*, Bz_s\rangle],
\end{aligned}$$

and hence

$$0 \leq F(z_s, y) + G(z_s, y) + (1-s)\langle Bz_s, y - x^*\rangle. \quad (3.16)$$

Letting $s \to 0^+$ in (3.16), we have, for each y in C,

$$0 \leq F(x^*, y) + G(x^*, y) + \langle y - x^*, Bx^*\rangle, \quad \forall y \in C$$

That is, $x^* \in EP(F, G, B)$.

Further, we show that x^* is also in $(A+R)^{-1}0$. Let $v \in Ru$. Setting

$$u_n = J_\lambda^R((1-t_n)x_n - \lambda Ax_n), \quad \forall n = 1, 2, \ldots,$$

we have

$$(1-t_n)x_n - \lambda Ax_n \in (I + \lambda R)u_n \Rightarrow \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{u_n}{\lambda} \in Ru_n.$$

Since R is monotone, we have, for $(u,v)\in R$,

$$
\begin{aligned}
& \Big\langle \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{u_n}{\lambda} - v, u_n - u \Big\rangle \ge 0 \\
\Rightarrow \quad & \langle (1-t_n)x_n - \lambda Ax_n - u_n - \lambda v, u_n - u\rangle \ge 0 \\
\Rightarrow \quad & \langle Ax_n + v, u_n - u\rangle \le \frac{1}{\lambda}\langle x_n - u_n, u_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, u_n - u\rangle \\
\Rightarrow \quad & \langle Ax^* + v, u_n - u\rangle \le \frac{1}{\lambda}\langle x_n - u_n, u_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, u_n - u\rangle + \langle Ax^* - Ax_n, u_n - u\rangle \\
& \qquad \le \frac{1}{\lambda}\|x_n - u_n\|\|u_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|u_n - u\| + \|Ax^* - Ax_n\|\|u_n - u\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\langle Ax^* + v, x^* - u\rangle \le \; & \frac{1}{\lambda}\|x_n - u_n\|\|u_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|u_n - u\| \\
& + \|Ax^* - Ax_n\|\|u_n - u\| + \langle Ax^* + v, x^* - u_n\rangle. \qquad (3.17)
\end{aligned}
$$

Since $\langle x_n - x^*, Ax_n - Ax^*\rangle \ge \alpha\|Ax_n - Ax^*\|^2$, $Ax_n \to Az$ and $x_n \rightharpoonup x^*$, we have

$$Ax_n \to Ax^*. \qquad (3.18)$$

We also observe that $t_n \to 0$, $\|x_n - u_n\| \to 0$ (Step 1), and thus $u_n \rightharpoonup x^*$. From (3.17), we derive

$$\langle -Ax^* - v, x^* - u\rangle \ge 0.$$

Since R is maximal monotone, we have $-Ax^* \in Rx^*$. This shows that $0 \in (A+R)x^*$. Hence, we have $x^* \in EP(F,G,B)\cap (A+R)^{-1}0 = \Gamma$.

At this moment, we can substitute x^* for z in (3.12) to get

$$\|x_n - x^*\|^2 \le -\Big\langle x^*, x_n - \frac{\lambda}{1-t_n}(Ax_n - Ax^*) - x^*\Big\rangle + t_nM.$$

Consequently, the weak convergence of $\{x_n\}$ to x^* actually implies that $x_n \to x^*$ by (3.18).

Finally, we return to (3.12) again and take the limit as $n\to\infty$ to get

$$\|x^* - z\|^2 \le -\langle z, x^* - z\rangle, \quad \forall z \in \Gamma.$$

Equivalently,

$$\|x^*\|^2 \le \langle x^*, z\rangle, \quad \forall z\in\Gamma. \qquad (3.19)$$

This clearly implies that

$$\|x^*\| \le \|z\|, \quad \forall z \in \Gamma.$$

This together with (3.19) implies that x^* is the unique element of Γ of minimum norm.

If x' is an other weak cluster point of the net $\{x_t\}$ as $t\to 0^+$, we will also see that x' is in Γ with $\|x'\| = \|x^*\|$. Then (3.19) implies $x' = x^*$. This ensures that $x_t \to x$ in norm as $t \to 0^+$. □

Theorem 3.2. *Let $y_0 \in C$. Define*

$$y_{n+1} = T_\mu(I - \mu B)J_\lambda^R((1-\alpha_n)I - \lambda A)y_n, \quad \forall n = 1, 2, \ldots. \tag{3.20}$$

Here, $\{\alpha_n\}$ is a sequence in $(0, 1 - \frac{\lambda}{2\alpha})$ satisfying conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = +\infty$*;*

(ii) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$*.*

Then the sequence $\{y_n\}$ converges strongly to the unique point x^* in Γ of minimum norm.

Proof. **Step 1.** Arguing in parallel to the proof of Theorem 3.1, we first show that both $\{y_n\}$ and $\{Ay_n\}$ are uniformly bounded, and

$$\lim_{n\to\infty} \|Ay_n - Az\| = 0, \quad \forall z \in \Gamma.$$

Take any z in Γ. Note that $J_\lambda^B(z - \lambda Az) = J_\lambda^B\Big(\alpha_n z + (1-\alpha_n)(z - \lambda Az/(1-\alpha_n))\Big)$ for all $n = 1, 2, \ldots$. As in deriving (3.7), we have

$$\begin{aligned} &\Big\|J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big) - z\Big\|^2 \\ \leq\ & (1-\alpha_n)\Big(\|y_n - z\|^2 + \frac{\lambda(\lambda - 2(1-\alpha_n)\alpha)}{(1-\alpha_n)^2}\|Ay_n - Az\|^2\Big) + \alpha_n\|z\|^2 \qquad (3.21) \\ \leq\ & (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z\|^2. \end{aligned}$$

From (3.20) and (3.21), we have

$$\begin{aligned} \|y_{n+1} - z\|^2 &= \|T_\mu(I-\mu B)J_\lambda^R((1-\alpha_n)I - \lambda A)y_n - T_\mu(I-\mu B)z\|^2 \\ &\leq \|J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big) - z\|^2 \qquad (3.22) \\ &\leq (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z\|^2. \\ &\leq \max\{\|y_n - z\|^2, \|z\|^2\}. \end{aligned}$$

By induction

$$\|y_n - z\| \leq \max\{\|y_0 - z\|, \|z\|\}.$$

Therefore, $\{y_n\}$ is bounded. Since A is α-inverse strongly monotone, it is $\frac{1}{\alpha}$-Lipschitz continuous. We deduce immediately that $\{Ay_n\}$ is also bounded.

From (3.20) and Lemma 2.2, we have

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ =\ & \|T_\mu(I-\mu B)J_\lambda^R((1-\alpha_n)I - \lambda A)y_n - T_\mu(I-\mu B)J_\lambda^R((1-\alpha_{n-1})I - \lambda A)y_{n-1}\| \\ \leq\ & \|((1-\alpha_n)I - \lambda A)y_n - ((1-\alpha_{n-1})I - \lambda A)y_{n-1}\| \\ =\ & \|(1-\alpha_n)(I - \lambda A/(1-\alpha_n))y_n - (1-\alpha_n)(I - \lambda A/(1-\alpha_n))y_{n-1} + (\alpha_{n-1} - \alpha_n)y_{n-1}\| \\ \leq\ & (1-\alpha_n)\|(I - \lambda A/(1-\alpha_n))y_n - (1-\alpha_n)(I - \lambda A/(1-\alpha_n))y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|y_{n-1}\| \\ \leq\ & (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\sup_k \|y_k\|. \end{aligned}$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.23}$$

By (3.21) and (3.22), we obtain

$$\|y_{n+1} - z\|^2 \leq (1-\alpha_n)\|y_n - z\|^2 + \frac{\lambda}{(1-\alpha_n)}(\lambda - 2(1-\alpha_n)\alpha)\|Ay_n - Az\|^2 + \alpha_n\|z\|^2.$$

Therefore,

$$\begin{aligned} & \frac{\lambda}{(1-\alpha_n)}(2(1-\alpha_n)\alpha - \lambda)\|Ay_n - Az\|^2 \\ \leq\ & \|y_n - z\|^2 - \|y_{n+1} - z\|^2 + \alpha_n(\|z\|^2 - \|y_n - z\|^2) \\ \leq\ & \|y_{n+1} - y_n\|(\|y_n - z\| + \|y_{n+1} - z\|) + \alpha_n(\|z\|^2 - \|y_n - z\|^2) \\ \to\ & 0. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|Ay_n - Az\| = 0.$$

Step 2. Next, we verify

$$\lim_{n\to\infty} \Big\| y_n - J_\lambda^R((1-\alpha_n)I - \lambda A)y_n \Big\| = 0.$$

Using (3.22) and as deriving (3.10) in the proof of Theorem 3.1, we have

$$\begin{aligned} \|y_{n+1} - z\|^2 \ \leq\ & \Big\| J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big) - z \Big\|^2 \\ \leq\ & (1-\alpha_n)\|y_n - z\|^2 + \alpha_n\|z\|^2 - \Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\|^2 \\ & +2\lambda\Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\|\|Ay_n - Az\|. \end{aligned}$$

Hence,

$$\begin{aligned} & \Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\|^2 \\ \leq\ & \|y_n - z\|^2 - \|y_{n+1} - z\|^2 + \alpha_n(\|z\|^2 - \|y_n - z\|^2) \\ & +2\lambda\Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\|\|Ay_n - Az\| \\ \leq\ & (\|y_n - z\| + \|y_{n+1} - z\|)\|y_{n+1} - y_n\| + \alpha_n(\|z\|^2 - \|y_n - z\|^2) \\ & +2\lambda\Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\|\|Ay_n - Az\|. \end{aligned}$$

Since $\alpha_n \to 0$, $\|y_{n+1} - y_n\| \to 0$ by (3.23) and $\|Ay_n - Az\| \to 0$ by Step 1, we deduce

$$\lim_{n\to\infty} \Big\|(1-\alpha_n)y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda Ay_n\Big)\Big\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\left\|y_n - J_\lambda^R\Big((1-\alpha_n)y_n - \lambda A y_n\Big)\right\| = 0.$$

Step 3. We next show that

$$\lim_{n\to\infty}\|y_n - T_\mu(I-\mu B)y_n\| = 0.$$

Set

$$u_n = J_\lambda^R((1-\alpha_n)y_n - \lambda A y_n), \quad \forall n = 1, 2, \ldots.$$

Then

$$y_{n+1} = T_\mu(I-\mu B)u_n, \quad \forall n = 1, 2, \ldots.$$

It follows from Step 2 that

$$\begin{aligned}\|y_{n+1} - T_\mu(I-\mu B)y_n\| &= \|T_\mu(I-\mu B)u_n - T_\mu(I-\mu B)y_n\| \\ &\le \|u_n - y_n\| \to 0.\end{aligned}$$

By (3.23) we see that

$$\|y_n - T_\mu(I-\mu B)y_n\| \to 0.$$

Step 4. We show that $y_n \to x^*$ which is the unique element in Γ with minimum norm (as given in Theorem 3.1).

Let $\{y_{n_k}\}$ be a subsequence of the bounded sequence $\{y_n\}$ weakly converging to some $\tilde{x}$ in C. By a similar argument as that of Step 4 in the proof of Theorem 3.1, we can show that $\tilde{x} \in \Gamma$. It follows from Theorem 3.1 that

$$\lim_{k\to\infty}\langle x^*, y_{n_k} - x^*\rangle = \langle x^*, \tilde{x} - x^*\rangle = \langle x^*, \tilde{x}\rangle - \|x^*\|^2 \ge 0.$$

As this holds true for all weakly convergent subsequences of $\{y_n\}$, we can conclude that $\liminf_{n\to\infty}\langle x^*, y_n - x^*\rangle \ge 0$. Since $y_n - u_n \to 0$ by Step 2, we also have

$$\liminf_{k\to\infty}\langle x^*, u_n - x^*\rangle \ge 0. \tag{3.24}$$

Using the firmly nonexpansivity of J_λ^R and the nonexpansivity of $I - \frac{\lambda}{1-\alpha_n}A$, we get

$$\begin{aligned}&\|u_n - x^*\|^2 \\ =\;& \left\|J_\lambda^R\Big((1-\alpha_n)y_n - \lambda A y_n\Big) - J_\lambda^R(x^* - \lambda A x^*)\right\|^2 \\ \le\;& \langle (1-\alpha_n)y_n - \lambda A y_n - (x^* - \lambda A x^*), u_n - x^*\rangle \\ =\;& (1-\alpha_n)\langle y_n - \lambda A y_n/(1-\alpha_n) - (x^* - \lambda A x^*/(1-\alpha_n)), u_n - x^*\rangle - \alpha_n\langle x^*, u_n - x^*\rangle \\ \le\;& (1-\alpha_n)\|y_n - \lambda A y_n/(1-\alpha_n) - (x^* - \lambda A x^*/(1-\alpha_n))\|\|u_n - x^*\| - \alpha_n\langle x^*, u_n - x^*\rangle \\ \le\;& (1-\alpha_n)\|y_n - x^*\|\|u_n - x^*\| - \alpha_n\langle x^*, u_n - x^*\rangle \\ \le\;& \frac{1-\alpha_n}{2}\|y_n - x^*\|^2 + \frac{1}{2}\|u_n - x^*\|^2 - \alpha_n\langle x^*, u_n - x^*\rangle.\end{aligned}$$

It follows that

$$\|u_n - x^*\|^2 \le (1-\alpha_n)\|y_n - x^*\|^2 - 2\alpha_n\langle x^*, u_n - x^*\rangle.$$

Therefore, by (3.22) we have

$$\begin{aligned}\|y_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\ &\le (1-\alpha_n)\|y_n - x^*\|^2 - 2\alpha_n\langle x^*, u_n - x^*\rangle.\end{aligned}$$

Applying Lemma 2.3 with (3.24) to the last inequality, we deduce $y_n \to x^*$ as asserted. □

Remark 3.3. We note that in Theorems 3.1 and 3.2, we add an additional assumption: the effective domain of R is included in C. This assumption is indeed not restrictive. The readers can find an example which satisfies this assumption in [25].

Remark 3.4. In Theorem 3.1, we have proved the facts that $\lim_{t\to 0+}\|Ax_t - Az\| = 0, \forall z \in \Gamma$(by (3.4)) and $\lim_{t\to 0+} x_t = x^*$. Thus, we deduce immediately that the image of Γ under A consists of exactly one point Ax^*, that is, $A(\Gamma) = \{Ax^*\}$.

This fact brings us a question: whether or not $A(\Gamma) = \{Ax^*\}$ implies Γ is a singleton set.

The answer is no. We give here an example to clarify this point. Let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $\mathrm{Fix}(T)$. It is easy to see that $I - T$ is monotone and Lipschitzian. If we take $A = I - T$ and $\Gamma = \mathrm{Fix}(T)$, then $A(\Gamma) = \{Ax^*\} = \{0\}$ for any fixed point x^* of T. However, $\mathrm{Fix}(T)$ can contain more than one points, in general.

We end this paper with an example showing that the set

$$\Gamma = EP(F, G, B) \cap (A + R)^{-1}0.$$

can be nonempty in our setting.

Example 3.5. Let $C = H = \mathbb{R}$. Let $F, G : C \times C \to \mathbb{R}$ be defined by

$$F(x, y) = x - y \quad \text{and} \quad G(x, y) = 0, \quad \forall x, y \in C.$$

Let $A, B : C \to H$ be defined by

$$Ax = -1 \quad \text{and} \quad Bx = \max\{0, x\}, \quad \forall x \in C.$$

Define $R : H \to 2^H$ by

$$Rx = \{x\}, \quad \forall x \in H.$$

Clearly, F, G satisfy conditions (F1)–(F4) and (G1)–(G3). For condition (H), we can set $a = x$ and $K = [x, x + \mu]$. On the other hand, by setting $\alpha = \beta = 1$, we see that A, B are α, β-inverse strongly-monotone from C into H, respectively. The effective domain of the maximal monotone operator R is C. So all the assumptions on A, B, F, G and R in Theorems 3.1 and 3.2 are satisfied.

In this case, it is easy to see that $EP(F,G,B) = \{1\}$ and

$$(A+R)^{-1}0 = \{x \in C : 0 \in Ax + Rx\} = \{x \in C : 0 \in \{-1+x\}\} = \{1\}.$$

Hence, $\Gamma = EP(F,G,B) \cap (A+R)^{-1}0 = \{1\}$ is nonempty.

Let $\lambda, \mu \in (0,2)$. It is straightforward to obtain

$$J_\lambda^R x = x/(1+\lambda) \quad \text{and} \quad T_\mu x = x + \mu, \quad \forall x \in H, x \in C.$$

Direct computation gives

$$x_t = \frac{\lambda+\mu}{\lambda+\mu+t(1-\mu)}, \quad \forall t \in (0, 1-\lambda/2),$$

as given in Theorem 3.1. As $t \to 0^+$, we see that $x_t \to 1$, which is the unique point in Γ.

Similarly, with the positive null sequence $\{\alpha_n\}$ given in Theorem 3.2, we have

$$y_{n+1} = \frac{(1-\alpha_n)y_n + \lambda}{1+\lambda}(1-\mu) + \mu, \quad \forall n = 0, 1, 2, \ldots.$$

It is not difficult to see that $y_n \to 1$, the unique point in Γ.

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