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Triple homomorphisms of C*-algebras

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In memory of our beloved friend, Kosita Beidar.

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Abstract. In this note, we will discuss what kind of operators between C*-algebras preserves Jordan triple products $\{a, b, c\} = (ab^*c + cb^*a)/2$. These include especially isometries and disjointness preserving operators.

Keywords: C*-algebras, Jordan triples, isometries, disjointness preserving operators.

1. Introduction

Recall that a Banach algebra A is an algebra with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$, and every Cauchy sequence converges. A complex Banach algebra A is a C*-algebra if there is an involution * defined on A such that $\|a^*a\| = \|a\|^2$. A special example is $B(H)$, the algebra of all bounded linear operators on a (complex) Hilbert space H. By the Gelfand-Naimark-Sakai Theorem, C*-algebras are exactly those norm closed *-subalgebras of $B(H)$. An abelian C*-algebra A can also be represented as the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X vanishing at infinity. X is compact if and only if A is unital.

It is well known that the algebraic structure determines the geometric (norm) structure of a C*-algebra A. Indeed, the norm of a self-adjoint element a of A coincides with the spectral radius of a, and the latter is a pure algebraic object. In general, the norm of an arbitrary element a of A is equal to $\|a^*a\|^{1/2}$, and a^*a is self-adjoint. For an abelian C*-algebra $A = C_0(X)$, we note that the underlying space X can be considered as the maximal ideal space of A consisting of complex homomorphisms (= linear and multiplicative functionals) of A. The topology of X is the hull-kernel topology, and thus be solely determined by the algebraic structure of A.

In this note, we will discuss how much the algebraic structure can be recovered if we know the norm, or other, structure of a C*-algebra. In particular, isometries and disjointness preserving operators of C*-algebras preserve triple products $\{a, b, c\} = (ab^*c + cb^*a)/2$.

The author is very grateful to our late friend, Kosita Beidar, from whom he learnt how to look at a seemingly pure analytic problem from the point of view of an algebraist.

2. The geometric structure determines the algebraic structure

Suppose $T : A \longrightarrow B$ is an isometric linear embedding between C*-algebras. That is, $\|Tx\| = \|x\|$ for all x in A. We are interested in knowing what kind of algebraic structure T inherits from A to its range, which is in general just a Banach subspace of B. We begin with two famous results.

Theorem 2.1. (Banach and Stone; see, e.g., [5]) *Let X and Y be locally compact Hausdorff spaces. Let $T : C_0(X) \longrightarrow C_0(Y)$ be a surjective linear isometry. Then T is a weighted composition operator*

$$Tf = h \cdot f \circ \varphi, \quad \forall f \in C_0(X),$$

where h is a continuous scalar function on Y with $|h(y)| \equiv 1$, and φ is a homeomorphism from Y onto X. Consequently, two abelian C-algebras are isomorphic as Banach spaces if, and only if, they are isomorphic as *-algebras.*

Here is a sketch of the proof. Let $T^* : M(Y) \longrightarrow M(X)$ be the dual map of T, which is again a surjective linear isometry from the Banach space $M(Y) = C_0(Y)^*$ of all bounded Radon measures on Y onto that on X. Restricting T^* to the dual unit balls, which are weak* compact and convex, we get an affine homeomorphism. Since the extreme points of the dual unit balls are exactly unimodular scalar multiples of point masses together with zero, T^* sends a point mass δ_y to $\lambda\delta_x$. Here $y \in Y$, $x \in X$ and $|\lambda| = 1$. We write $x = \varphi(y)$ and $\lambda = h(y)$ to indicate that x and λ depend on y. It follows that

$$Tf(y) = T^*(\delta_y)(f) = h(y)\delta_{\varphi(y)}(f) = h(y)f(\varphi(y)).$$

In other words, $Tf = h \cdot f \circ \varphi$, $\forall f \in C_0(X)$. It is then routine to see that h is unimodular and continuous on Y, and that φ is a homeomorphism from Y onto X.

Theorem 2.2. (Kadison [6]) *Let A and B be C*-algebras. Let $T : A \longrightarrow B$ be a surjective linear isometry. Then there is a unitary element u in $\tilde{B} = B \oplus \mathbb{C}1$, the unitalization of B, and a Jordan *-isomorphism $J : A \longrightarrow B$ such that*

$$Ta = uJ(a), \quad \forall a \in A.$$

Consequently, two C-algebras are isomorphic as Banach spaces if, and only if, they are isomorphic as Jordan *-algebras.*

Recall that a Jordan *-isomorphism J preserves linear sums, involutions and Jordan products: $a \circ b = (ab + ba)/2$. It is easy to see that the abelian case can also be written in this form with $u = h$ and $Jf = f \circ \varphi$. In general, the product of a pair of elements in A can be decomposed into two parts $ab = a \circ b + [a, b]$, the sum of the Jordan product and the Lie product $[a, b] = (ab - ba)/2$. It is plain that $a \circ b = b \circ a$ is commutative and $[a, b] = -[b, a]$ is anti-commutative. However they are not associative. The Kadison theorem states that the norm structure of a C*-algebra determines completely its Jordan structure.

It is interesting to note that Jordan products are determined by squares:

$$a \circ b = \frac{(a+b)^2 - a^2 - b^2}{2}, \quad \forall a, b \in A.$$

A similar algebraic structure exists in C*-algebras, namely, the Jordan triple products:

$$\{a, b, c\} = \frac{ab^*c + cb^*a}{2}.$$

There is also a polar identity for triples:

$$\{a, b, c\} = \frac{1}{8} \sum_{\alpha^2=1} \sum_{\beta^4=1} \alpha\beta \{a + \alpha b + \beta c\}^{(3)},$$

Hence, a linear map T between C*-algebras preserves triple products if and only if it preserves cubes $a^{(3)} = \{a, a, a\} = aa^*a$.

Kaup [7] rephrased Kadison theorem: a linear surjection between C*-algebras $T : A \longrightarrow B$ is an isometry if and only if it preserves triple products. A geometric proof of the Kadison Theorem is given by Dang, Friedman and Russo [2]. It goes first to note that a norm exposed face of the dual unit ball U_{B^*} is of the form $F_u = \{\varphi \in B^* : \|\varphi\| = \varphi(u) \leq 1\}$ for a unique partial isometry u in B^{**}. For two φ, ψ in B^*, they are said to be orthogonal to each other if they have polar decompositions $\varphi = u|\varphi|, \psi = v|\psi|$ such that $u \perp v$, i.e., $u^*v = uv^* = 0$. This amounts to say that $\|\varphi \pm \psi\| = \|\varphi\| + \|\psi\|$. Two faces F_u, F_v are orthogonal if and only if $u \perp v$. Then they verify that the adjoint T^* of the surjective linear isometry T maps faces to faces and preserves orthogonality. Consequently, T sends orthogonal partial isometries to orthogonal partial isometries. By the spectral theory, every element a in A can be approximated in norm by a finite linear sum of orthogonal partial isometries $\sum_j \lambda_j u_j$. Then its cube $a^{(3)}$ can also be approximated by $\sum_j \lambda_j^{(3)} u_j$. It follows that $T(a^{(3)})$ and $(Ta)^{(3)}$ can both be approximated by $\sum_j \lambda_j^{(3)} Tu_j$. Hence $T(a^{(3)}) = (Ta)^{(3)}$, and thus T preserves triple products by the polar identity.

We note that the above (geometric) proof of the Kadison theorem quite depends on the fact the range of the isometry is again a C*-algebra. Extending the Holsztynski theorem [3, 5], Chu and Wong [1] studied non-surjective linear isometries between C*-algebras.

Theorem 2.3. (Chu and Wong [1]) *Let A and B be C*-algebras and let T be a linear isometry from A* into *B. There is a largest closed projection p in B^{**} such that $T(\cdot)p : A \longrightarrow B^{**}$ is a Jordan triple homomorphism and*

$$T(ab^*c + cb^*a)p = T(a)T(b)^*T(c)p + T(c)T(b)^*T(a)p, \quad \forall a, b, c \in A.$$

When A is abelian, we have $\|T(a)p\| = \|a\|$ for all a in A. In particular, T reduces locally to a Jordan triple isomorphism on the JB-triple generated by any a in A, by a closed projection p_a.*

Beside the triple technique, the proof of above theorem also makes use of the concept of representing elements in a C*-algebra as special sections of a continuous field of Hilbert spaces developed in [8]. It is still geometric.

3. Disjointness preserving operators are triple homomorphisms

In this section, we do not assume the operator T is isometric. Although the following statement might have been known to experts, we provide a new and short proof here as we do not find any in the literature. For simplicity of notations, we also write T for its bidual map $T^{**} : A^{**} \longrightarrow B^{**}$.

Theorem 3.1. *Let $T : A \longrightarrow B$ be a bounded linear map between C*-algebras. Then T is a triple homomorphism if and only if T sends partial isometries to partial isometries.*

Proof. One direction is trivial. Suppose T sends partial isometries to partial isometries. Let u, v be two partial isometries in A. Observe that they are orthogonal to each other, namely, $u^*v = uv^* = 0$, if and only if they have orthogonal initial spaces and orthogonal range spaces. This amounts to say that $u + \lambda v$ is a partial isometry for all scalar λ with $|\lambda| = 1$. Consequently, T sends orthogonal partial isometries to orthogonal partial isometries. For every a in A, approximate a in norm by a finite linear sum $\sum_n \lambda_n u_n$ of orthogonal partial isometries. Then its cube $a^{(3)} = aa^*a$ can also be approximated in norm by $\sum_n \lambda_n^{(3)} u_n$. It follows that Ta and $T(a^{(3)})$ can be approximated in norm by $\sum_n \lambda_n T u_n$ and $\sum_n {\lambda_n}^{(3)} T u_n$, respectively. This gives $T(a^{(3)}) = (Ta)^{(3)}$, $\forall a \in A$. By the polar identity, we see that T is a triple homomorphism. ∎

We say that a linear map $T : A \longrightarrow B$ between C*-algebras is *disjointness preserving* if

$$a^*b = ab^* = 0 \quad \text{implies} \quad (Ta)^*(Tb) = (Ta)(Tb)^* = 0, \quad \forall a, b \in A.$$

Clearly, T is disjointness preserving if and only if it preserves disjointness of partial isometries. It is clear that every triple homomorphism preserves disjointness. Looking at the well-known abelian case, that is, the Jarosz theorem [4, 5], we see that not every disjointness preserving map is a triple homomorphism. Indeed, let $T : C_0(X) \longrightarrow C_0(Y)$ be a bounded disjointness preserving linear map between abelian C*-algebras. Then there is a closed subset Y_0 of Y on which every Tf vanishes. On $Y_1 = Y \setminus Y_0$ there is a bounded continuous function h and a continuous map φ from Y_1 into X such that $Tf_{|Y_1} = h \cdot f \circ \varphi$ for all f in $C_0(X)$. Hence, T is a triple homomorphism if and only if $T1$ is a partial isometry in $C_0(Y)^{**}$. We end this note with a proof of this fact for the non-abelian case.

Theorem 3.2. *Let $T : A \longrightarrow B$ be a bounded linear map between C*-algebras. Then T is a triple homomorphism if and only if T is disjointness preserving and $T1$ is a partial isometry.*

Proof. We verify the sufficiency only. By the polar identity it suffices to check that T sends the cube $a^{(3)}$ to the cube $(Ta)^{(3)}$ for every element a of A. Identify the JB*-triple of A generated by 1 and a with $C(X)$ (see [7, Corollary 1.15]), where X is some compact set of complex numbers. Denote again by T the bidual map of T from $C(X)^{**}$ into B^{**}.

Let $X = \cup_n X_n$ be any finite Borel partition of X and pick an arbitrary point x_n from X_n. In particular,

$$1 = \sum_n 1_{X_n},$$

where 1_{X_n} is the characteristic function of the Borel set X_n. For $j \neq k$, we can find two sequences $\{f_m\}_m$ and $\{g_m\}_m$ in $C(X)$ such that $f_{m+p}g_m = 0$ for $m, p = 0, 1, \ldots$, $f_m \to 1_{X_j}$ and $g_m \to 1_{X_k}$ pointwisely on X. By the weak* continuity of T, we see that

$$T(1_{X_j})T(g_m)^* = \lim_{p\to\infty} T(f_{m+p})T(g_m)^* = 0 \quad \text{for all } m = 1, 2, \ldots.$$

Thus

$$T(1_{X_j})T(1_{X_k})^* = \lim_{m\to\infty} T(1_{X_j})T(g_m)^* = 0.$$

Similarly, we have

$$T(1_{X_j})^*T(1_{X_k}) = 0.$$

Consequently, for each j we have

$$T(1)T(1_{X_j})^*T(1) = \sum_{m,n} T(1_{X_n})T(1_{X_j})^*T(1_{X_m}) = (T(1_{X_j}))^{(3)}.$$

This gives

$$\sum_n T(1_{X_n}) = T1 = (T1)^{(3)} = \sum_n (T(1_{X_n}))^{(3)}.$$

Multiplying the above identity on the left by $T(1_{X_n}))^*$ and $(T(1_{X_n}))^{(3)^*}$ respectively, we see that

$$(T(1_{X_n}) - (T(1_{X_n}))^{(3)})^*(T(1_{X_n}) - (T(1_{X_n}))^{(3)}) = 0.$$

Hence $T(1_{X_n})$ is a partial isometry for each n and orthogonal to the others. It follows that

$$\begin{aligned}(T(f))^{(3)} &= \lim \left(\sum_n f(x_n)T(1_{X_n})\right)^{(3)} = \lim \sum f(x_n)^{(3)}(T(1_{X_n}))^{(3)} \\ &= \lim \sum f(x_n)^{(3)}T(1_{X_n}) = T(f^{(3)}),\end{aligned}$$

for all f in $C(X)$. This completes the proof. ∎

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