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The solution of the dynamic-coupling partial differential equations with decomposition of a generalized function δ

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Abstract

If the solutions of a dynamic partial differential equation are couple, there are lots of troubles to deal with it. A process on decouple of fast and slow dilational waves to solve the Biot's two-phase dynamic partial differential equation with decomposition the δ function was been shown in this paper. There are the results of solving equation in paper, these results are compared with previous solutions. The situation of good agreement of both is not only to express that the results obtained in this paper are right, and indicate that a performing decomposition δ function may be effective and convenient to solve the partial differential equation sometime.

Kay words

Solution, dynamic-coupling, partial differential, decomposition, generalized function δ

Introduction

There are lots of achievements obtained by dint of a generalized function δ to solve the dynamic partial differential equation [1--3]. However, if the solutions of a dynamic partial differential equation are coupling, then we may face a lot of troubles when we use this δ function to deal with [4, 5]. In this aspect, Biot's dynamic equations of a two-phase saturated medium are the most typical [6, 7]. Biot's dynamic equations of a two-phase saturated medium play an important role in soil dynamics [8—11], seismic engineering [12, 13], and geophysics [14—16]. Because of the existence and coupling of two dilational waves on solutions of Biot's dynamic equations of a two-phase saturated medium, the ordinary potential decomposition is inability, so that Chen.J attained the results based on the continuation of the solution of partial differential equation with δ inhomogenous term and the discontinuous of the first-order derivation of that until the end of last century [4, 5]. Chen's solution is extremely complex. If we can decompose the δ function in Biot's two-phase dynamic partial differential equation, we can decouple fast and slow dilational waves also [17, 18], then make use of potential decomposition to solve the problems conveniently.

1. The solutions on Biot's two-phase dynamic partial differential equations

1.1 the ordinary solution

Biot's equation can be written as follows [4, 5]

$$\begin{cases}(\lambda_c+2\mu)u_{j,ij}+\mu\delta_{ikh}\delta_{hlj}u_{i,lk}+\alpha Mw_{j,ij}+f_i=\rho\ddot{u}_i+\rho_f\ddot{w}_i\\ \alpha Mu_{j,ij}+Mw_{j,ij}=\rho_f\ddot{u}_i+\gamma(\omega)\ddot{w}_i\end{cases} \tag{1}$$

where $\lambda_c,\mu,\alpha.M$ are the mechanics parameters, u_i is the displacement of solid skeleton in i direction, $(\ddot{u}_i=\mathrm{d}^2u/\mathrm{d}t^2)$ w_i is the average displacement of the fluid-phase relative to the solid-phase in i direction($w_i=\beta_0(U_i-u_i)$ with U_i denoting the average displacement of fluid, $\ddot{w}_i=\mathrm{d}^2w/\mathrm{d}t^2$). ρ , ρ_f are the material density of the two-phase and the flow-phase respectively; δ_{ikm} is the permutation tensor. The form of Laplace transmit on Eqn.(1) is:

$$\left(\lambda+\mu\right)\tilde{u}_{j,ij}+\mu\tilde{u}_{j,jj}\eta_1\tilde{p}_j-\rho_1 s^2\tilde{u}_i+\tilde{f}_i=0 \tag{2}$$

$$\zeta\tilde{p}_{ij}-\frac{s}{M}\tilde{p}-\eta_2 s\tilde{u}_{i,i}+\tilde{\gamma}=0 \tag{3}$$

Eqn.（3）is got from Darcy's law, where $i,j=1,2,3$,the tilde denotes the Laplace transformation, $\eta_1=\alpha-\rho_f s\zeta$, $\eta_2=\alpha-\rho_f s\zeta$, $\rho_2=\rho-\rho_f^2 s\zeta$, $\zeta=\left(\left(1/\kappa\right)+ms\right)^{-1}$, κ is the permeability, m is the fluid of a unit volume, and s is the Laplace transform parameter. Equation(2) and (3) are nondimensionalized by using the parameters [4, 5]:

$$\xi_i = \frac{x}{\rho\kappa\alpha_1} \quad \text{and} \quad \tau = \frac{t}{\rho\kappa} \tag{4}$$

where α_1 is the velocity of fast dilatinal wave, we define a dimensionless displacement and pore pressure through, next:

$$U_i = \frac{u_i}{\rho\kappa\alpha_1}, \quad P = \frac{p}{\rho\alpha_1^2} \tag{5}$$

The nondimensional form for the field Eqn. (2) and the energy Eqn.(3) is then

$$(\lambda * + \mu *)\tilde{U}_{j,ij} + \mu * \tilde{U}_{j,jj} - \eta_1 * \tilde{P}_{.i} - \rho_1 * s^2 \tilde{U}_i + \tilde{F}_i = 0 \tag{6}$$

$$\zeta * \tilde{P}_{.ii} - \frac{s}{M *}\tilde{P} - \eta_2 * s\tilde{U}_{i,i} + \tilde{\Gamma} = 0 \tag{7}$$

where $\tilde{F}_i$ and $\tilde{\Gamma}$ are nondimensionalized body force and fluid source injection, and

$$\lambda * = \frac{\lambda}{\lambda + 2\mu + \alpha^2 M}, \; \mu * = \frac{\mu}{\lambda + 2\mu + \alpha^2 M}, \; M * = \frac{M}{\lambda + 2\mu + \alpha^2 M}, \; \rho * = 1, \; \rho_f * = \frac{\rho_f}{\rho},$$

$$m * = \frac{m}{\rho}, \kappa * = 1, \; \rho_1 * = \rho - (\rho_f *)^2 s\zeta *, \; \zeta * = \frac{1}{m * s + 1/\kappa}$$

Equation (2) and (3) can be written as follows [4, 5]:

$$\mathbf{B}(\partial x, s)\tilde{\mathbf{U}} + \tilde{\mathbf{F}} = 0 \tag{8}$$

where：

$$B_{ij}(\partial x, s) = (\lambda + \mu)\frac{\partial^2}{\partial x_i \partial x_j} + \delta_{ij}(\mu\Delta - \rho_1 s^2), \quad B_{i3}(\partial x, s) = -c_1 \frac{\partial}{\partial x_i},$$

$$B_{3j}(\partial x, s) = -c_2 s \frac{\partial}{\partial x_j}, \quad B_{33}(\partial x, s) = \zeta\Delta - \frac{s}{Q}, \quad c_1 = c_2 = \alpha - \rho_f s\zeta$$

So Equation (8) is as [4, 5]:

$$\left[\det(\mathbf{B}(\partial x, s)\mathbf{I})\right]\varphi + \frac{1}{s}\mathbf{I}\delta(x) = 0 \tag{9}$$

Nondimensionalized Equation (8) can be expressed as [4, 5]

$$\mathbf{B}(\partial x, s)\mathbf{B}^*(\partial x, s)\varphi + \mathbf{I}\frac{1}{s}\delta(x) = 0 \tag{10}$$

For Eqn.(10), $c_1{*} = c_2{*} = \alpha - \rho_f * s\zeta *$. Consequently, we get [4, 5]:

$$\tilde{\mathbf{G}} = \mathbf{B}*\left(\partial x, s\right)\varphi \tag{11}$$

The solution in time domain of Eqn (11) with the inverse of Laplace transformation under a Heavide force is follows [4, 5]:

$$\begin{aligned}
G_{ij} = {} & \int_{r/\alpha_2}^{t} [P_{11}e^{-b(t-\tau)} + P_{12}e^{-c(t-\tau)} + P_{13}]e^{-\eta_\alpha \tau} I_0(\xi_\alpha \sqrt{\tau^2 - r^2/\alpha_2^2})\mathrm{d}\tau H(t - r/\alpha_2) \\
& + \int_{r/\alpha_2}^{t} [P_{21}e^{-a(t-\tau)} + P_{22}e^{-b(t-\tau)} + P_{23}e^{-c(t-\tau)} + P_{24} + P_{25}(t-\tau) + P_{26}(t-\tau)^2] \\
& \times e^{-\eta_\alpha \tau} \frac{\xi_\alpha \; r/\alpha_2}{\sqrt{\tau^2 - r^2/\alpha_2^2}} I_1(\xi_\alpha \sqrt{\tau^2 - r^2/\alpha_2^2})\mathrm{d}\tau H(t - r/\alpha_2) + [P_{21}e^{-a(t-r/\alpha_2)} + P_{22}e^{-b(t-r/\alpha_2)} \\
& + P_{23}e^{-c(t-r/\alpha_2)} + P_{24} + P_{25}(t - r/\alpha_2) + P_{26}(t - r/\alpha_2)^2]e^{-\eta_\alpha r/\alpha_2} H(t - r/\alpha_2) \\
& + \int_{r/\alpha_1}^{t} [P_{31}e^{-b(t-\tau)} + P_{32}e^{-c(t-\tau)} + P_{33}]e^{-\eta_\alpha \tau} I_0(\xi_\alpha \sqrt{\tau^2 - r^2/\alpha_1^2})\mathrm{d}\tau H(t - r/\alpha_1) \\
& + \int_{r/\alpha_1}^{t} [P_{41}e^{-a(t-\tau)} + P_{42}e^{-b(t-\tau)} + P_{43}e^{-c(t-\tau)} + P_{44} + P_{45}(t-\tau) + P_{46}(t-\tau)^2] \\
& \times e^{-\eta_\alpha \tau} \frac{\xi_\alpha \, r/\alpha_1}{\sqrt{\tau^2 - r^2/\alpha_1^2}} I_1(\xi_\alpha \sqrt{\tau^2 - r^2/\alpha_1^2})\mathrm{d}\tau H(t - r/\alpha_1) + [P_{41}e^{-a(t-r/\alpha_1)} + P_{42}e^{-b(t-r/\alpha_1)} \\
& + P_{43}e^{-c(t-r/\alpha_1)} + P_{44} + P_{45}(t - r/\alpha_1) + P_{46}(t - r/\alpha_1)^2]e^{-\eta_\alpha r/\alpha_1} H(t - r/\alpha_1) \\
& + \int_{r/\beta}^{t} P_{51}e^{-\eta_\beta \tau} I_0(\xi_\beta \sqrt{\tau^2 - r^2/\beta^2})\mathrm{d}\tau H(t - r/\alpha_2\beta) + \int_{r/\beta}^{t} [P_{61}e^{-a(t-\tau)} + P_{62} \\
& + P_{63}(t-\tau) + P_{64}(t-\tau)^2]e^{-\eta_\beta \tau} \frac{\xi_\beta \, r/\beta}{\sqrt{\tau^2 - r^2/\beta^2}} I_1(\xi_\beta \sqrt{\tau^2 - r^2/\beta^2})\mathrm{d}\tau H(t - r/\beta) \\
& + [P_{61}e^{-a(t-r/\beta)} + P_{62} + P_{63}(t - r/\beta) + P_{64}(t - r/\beta)^2]e^{-\eta_\beta r/\beta} H(t - r/\beta)
\end{aligned} \tag{12}$$

where I_0 and I_1 are the zero and first order of bassel's function respectively. η_α, η_β are the dissipation factor of dilational and distortional wave, respectively. α_2 and β are the velocity of slow dilational and distortional wave respectively. The meanings of the residual marks in Eqn.(12) can refer to Appendix 1

1.2 the decomposition of Generalized function δ

For Eqn (1), when u_i and w_i are components of the non-divergence field, we can derive the following[17]:

$$\begin{cases} \rho_f \ddot{u}_i + \gamma(\omega)\ddot{w}_i = 0 \\ w_i = -\rho_f u_i / \gamma(\omega) \end{cases} \tag{13}$$

For the non-curl field, the components u_i and w_i can be written as

$$\begin{cases} u_{1i} + u_{2i} = u_i \\ \xi_1 u_{1i} + \xi_2 u_{2i} = w_i \end{cases} \tag{14}$$

and

$$\xi_n = (\lambda_c + 2\mu - \rho\alpha_n^2)/(\rho_f \alpha_n^2 - \alpha M) \qquad (n = 1, 2) \tag{15}$$

Substituting Eqns. (13) and (14) into Eqn. (1), we obtain the following[17]:

$$\begin{aligned} &(\lambda_c + 2\mu + \alpha M \xi_1)u_{1j,ij} + (\lambda_c + 2\mu + \alpha M \xi_2)u_{2j,ij} + \mu\delta_{ikh}\delta_{hlj}u_{i,lk} \\ &= \begin{cases} (\rho + \rho_f \xi_1)\ddot{u}_{1i} + (\rho + \rho_f \xi_2)\ddot{u}_{2i} & \textit{(non-curl field)} \\ (\rho - \rho_f^2/\gamma(\omega))\ddot{u}_i & \textit{(non-divergence field)} \end{cases} \end{aligned} \tag{16}$$

Introduction a generalized function $\delta(\mathbf{r})$, $\delta(\mathbf{r}) = \delta(\mathbf{x} - \boldsymbol{\varsigma})$, x_i and ς_i are the respective components of $\mathbf{x}$ and $\boldsymbol{\varsigma}$, the coordinates of the field point and the source point, respectively. Thus, Eqn. (15) becomes

$$\begin{aligned} &\begin{cases} (\lambda_c + 2\mu + \alpha M \xi_1)u_{1j,ij} + (\lambda_c + 2\mu + \alpha M \xi_2)u_{2j,ij} + \mu\delta_{ikh}\delta_{hlj}u_{i,lk} \\ \alpha M u_{j,ij} + M w_{j,ij} = \rho_f \ddot{u}_i + \gamma(\omega)\ddot{w}_i \end{cases} \\ &\quad - \begin{cases} (\rho + \rho_f \xi_1)\ddot{u}_{1i} + (\rho + \rho_f \xi_2)\ddot{u}_{2i} \\ (\rho - \rho_f^2/\gamma(\omega))\ddot{u}_i \end{cases} = \delta(\mathbf{r}) \end{aligned} \tag{17}$$

The Fourier transform of Eqn. (17) is

$$\begin{aligned} &(\rho + \rho_f \xi_1)\alpha_1^2 \tilde{u}_{1j,ij} + (\rho + \rho_f \xi_2)\alpha_2^2 \tilde{u}_{2j,ij} + (\rho - \frac{\rho_f^2}{\gamma})\beta^2 \delta_{ikh}\delta_{hlj}\tilde{u}_{i,lk} \\ &\quad + \begin{cases} (\rho + \rho_f \xi_1)\omega^2 \tilde{u}_{1i} + (\rho + \rho_f \xi_2)\omega^2 \tilde{u}_{2i} \\ (\rho - \rho_f^2/\gamma(\omega))\omega^2 \tilde{u}_i \end{cases} = \delta(\mathbf{r}) \end{aligned} \tag{18}$$

For $\omega < \omega_c$, where ω_c is a cut off frequency $\omega_c = 0.06\pi k_d \rho_f / \eta_d \beta$, and low frequencies

$\gamma(\omega)=\gamma$, γ is constant[18]. Then we decompose Generalized function $\delta(\mathbf{r})$ to yield[17]:

$$\begin{aligned}\delta(\mathbf{r}) &= -\frac{1}{4\pi}(1/r)_{,ii} = -\frac{1}{4\pi}\left[(1/r)_{j,ij} - \delta_{ikh}\delta_{hlj}(1/r)_{i,lk}\right] \\ &= -\frac{1}{4\pi}\left[\frac{1+\xi_1}{\xi_1-\xi_2}(1/r)_{j,ij} - \frac{1+\xi_2}{\xi_1-\xi_2}(1/r)_{j,ij} - \delta_{ikh}\delta_{hlj}(1/r)_{i,lk}\right]\end{aligned} \tag{19}$$

where $r=[(x_i-\varsigma_i)(x_i-\varsigma_i)]^{1/2}$ is the distance between the source point and the field point.

Comparing Eqn. (19) with Eqn. (18), we find that

$$\left[\frac{1+\xi_1}{\xi_1-\xi_2}(1/r)_{j,j}\right] \text{ and } -\left[\frac{1+\xi_2}{\xi_1-\xi_2}(1/r)_{j,j}\right]$$

are two field potentials of the dilational wave in a two-phase saturated medium, caused by solid–fluid interaction. Equation (19) coincides with the compatibility of fast and slow dilational waves indicated by Biot (1956) and examined by Chen (1994). Substituting Eqn. (19) into Eqn. (18), we have;

$$\begin{aligned}&(\rho+\rho_f\xi_1)\alpha_1^2\tilde{u}_{1j,ij} + (\rho+\rho_f\xi_2)\alpha_2^2\tilde{u}_{2j,ij} + (\rho-\frac{\rho_f^2}{\gamma})\beta^2\delta_{ikh}\delta_{hlj}\tilde{u}_{i,lk} \\ &+\begin{cases}(\rho+\rho_f\xi_1)\omega^2\tilde{u}_{1i} + (\rho+\rho_f\xi_2)\omega^2\tilde{u}_{2i} \\ (\rho-\rho_f^2/\gamma)\omega^2\tilde{u}_i\end{cases} \\ &= -\frac{F_0\tilde{g}(\omega)}{4\pi}\left[\frac{1+\xi_1}{\xi_1-\xi_2}(1/r)_{j,ij} - \frac{1+\xi_2}{\xi_1-\xi_2}(1/r)_{i,ij} - \delta_{ikh}\delta_{hlj}(1/r)_{i,lk}\right]\cdot K_j\end{aligned} \tag{20}$$

where $K_{\alpha_1}=\omega/\alpha_1$, $K_{\alpha_2}=\omega/\alpha_2$, and $K_\beta=\omega/\beta$ are wave numbers of the fast and slow dilational waves, and distortional wave, respectively. We get the solution of Eqn.(1) easily.

$$\begin{aligned}\tilde{u}_i &= -\frac{1}{4\pi\omega^2}\left\{\frac{\lambda_1}{\rho+\rho_f\xi_1}(e^{iK_{\alpha 1}r}/r)_{j,ij} - \frac{\lambda_2}{\rho+\rho_f\xi_2}(e^{iK_{\alpha 2}r}/r)_{j,ij}\right. \\ &\left. -\frac{1}{\rho-\rho_f^2/\gamma}\delta_{ikh}\delta_{hlj}(e^{iK_\beta r}/r)_{i,lk}\right\}\cdot K_j = G_{ij}(\mathbf{x}\boldsymbol{\xi},\omega)\cdot e_j\end{aligned} \tag{21}$$

We define $G_{ij}(\mathbf{x},\xi,\omega)$ to be the second-order tensor of Green's function in the Fourier transform domain and e_j is the component of a unit force in the j^{th} direction [19].

$$G_{ij}(\mathbf{x},\xi,\omega) = -\frac{1}{4\pi\omega^2}\left\{\frac{\lambda_1}{\rho+\rho_f\xi_1}(e^{iK_{\alpha1}r}/r)_{j,ij} - \frac{\lambda_2}{\rho+\rho_f\xi_2}(e^{iK_{\alpha2}r}/r)_{j,ij} - \frac{1}{\rho-\rho_f^2/\gamma}\delta_{ikh}\delta_{hlj}(e^{iK_\beta r}/r)_{i,lk}\right\} \tag{22}$$

where $\lambda_1 = (1+\xi_1)/(\xi_1-\xi_2)$ and $\lambda_2 = (1+\xi_2)/(\xi_1-\xi_2)$.

Note that $(\delta_{mn}\varphi)_{,i} = \varphi_{,i}$ and $\delta_{kil}(\delta_{mn}\varphi)_{i,l} = \delta_{klm}(\varphi_{,i})_l\,\delta_{mn}$ [20]. The Green's function in the time-domain can be obtained using the inverse Fourier transformation as follows:

$$\begin{aligned}G_{ij}(\mathbf{x},\xi,t) = \frac{1}{4\pi}\Bigg\{&\frac{1}{r}r_{,i}r_{,j}\left[\frac{\lambda_1}{\rho+\rho_f\xi_1}\frac{1}{\alpha_1^2}g(t-r/\alpha_1) - \frac{1}{\rho-\rho_f^2/\gamma}\frac{1}{\beta^2}g(t-r/\beta)\right.\\ &\left.-\frac{\lambda_2}{\rho+\rho_f\xi_2}\frac{1}{\alpha_2^2}g(t-r/\alpha_2)\right] + \frac{1}{\rho-\rho_f^2/\gamma}\frac{1}{\beta^2 r}g(t-r/\beta)\delta_{ij} +\\ &+(1/r)_{,ij}\cdot\left[\frac{\lambda_1}{\rho+\rho_f\xi_1}\int_0^{r/\alpha_1}g(t-\tau)\tau d\tau - \frac{1}{\rho-\rho_f^2/\gamma}\int_0^{r/\beta}g(t-\tau)\tau d\tau - \frac{\lambda_2}{\rho+\rho_f\xi_2}\int_0^{r/\alpha_2}g(t-\tau)\tau d\tau\right]\Bigg\}\end{aligned} \tag{23}$$

If $g(t) = \delta(t)$ in Eqn. (23), we can obtain the following Green's function for a impulse:

$$\begin{aligned}G_{ij} = \frac{1}{4\pi}\Bigg\{&\frac{1}{r}\frac{x_i-\zeta_i}{r}\frac{x_j-\zeta_j}{r}\left[\frac{\lambda_1}{\rho+\rho_f\xi_1}\frac{1}{\alpha_1^2}\delta(t-r/\alpha_1) - \frac{\lambda_2}{\rho+\rho_f\xi_2}\frac{1}{\alpha_2^2}\delta(t-r/\alpha_2)\right.\\ &\left.-\frac{1}{\rho-\rho_f^2/\gamma}\frac{1}{\beta^2}\delta(t-r/\beta)\right] + \frac{\delta_{ij}}{\rho-\rho_f^2/\gamma}\frac{1}{\beta^2 r}\delta(t-r/\beta) + \left[-\frac{1}{r^3} + 3\frac{(x_i-\zeta_i)(x_j-\zeta_j)}{r^5}\right]\\ &\times\left[\frac{\lambda_1}{\rho+\rho_f\xi_1}tH(t-r/\alpha_1) - \frac{\lambda_2}{\rho+\rho_f\xi_2}tH(t-r/\alpha_2) - \frac{1}{\rho-\rho_f^2/\gamma}tH(t-r/\beta)\right]\Bigg\}\end{aligned} \tag{24}$$

For simplicity, the following notation is used:

$$D_1 = 1/(\rho+\rho_f\xi_1)\alpha_1^2,\quad D_2 = 1/(\rho+\rho_f\xi_2)\alpha_2^2\ ,\quad D_3 = 1/(\rho-\rho_f^2/\gamma(\omega))\beta^2$$

which represents the flexibility coefficients of fast and slow dilational waves, and the distortional wave, respectively. Replacing $x_i-\zeta_i$ with x_i, we can rewrite Eqn. (24) as

$$G_{ij}=\frac{1}{4\pi}\left\{\frac{x_ix_j}{r^3}\left[\lambda_1D_1\delta(t-\frac{r}{\alpha_1})-\lambda_2D_2\delta(t-\frac{r}{\alpha_2})-D_3\delta(t-\frac{r}{\beta})\right]+D_3\frac{\delta_{ij}}{r}\delta(t-\frac{r}{\beta})\right.$$
$$\left.+\left(-\frac{1}{r^3}+3\frac{x_ix_j}{r^5}\right)\left[\alpha_1^2\lambda_1D_1tH(t-\frac{r}{\alpha_1})-\alpha_2^2\lambda_2D_2tH(t-\frac{r}{\alpha_2})-\beta^2D_3tH(t-\frac{r}{\beta})\right]\right\} \quad (25)$$

1.3 the comparison of the results between two methods

Integrating Eqn.(25) or substituting $g(t)=H(t)$ to Eqn.(23), we can obtain the solution to the Green's function under Heaviside force, we have

$$G_{ij}=\frac{1}{4\pi}\left\{\frac{x_ix_j}{r^3}\left[\lambda_1D_1H(t-\frac{r}{\alpha_1})-\lambda_2D_2H(t-\frac{r}{\alpha_2})-D_3H(t-\frac{r}{\beta})\right]+D_3\frac{\delta_{ij}}{r}H(t-\frac{r}{\beta})\right.$$
$$\left.+3\frac{x_ix_j}{r^5}\left[\alpha_1^2\lambda_1D_1\frac{1}{2}(t^2-\frac{r^2}{\alpha_1^2})-\alpha_2^2\lambda_2D_2\frac{1}{2}(t^2-\frac{r^2}{\alpha_2^2})-\beta^2D_3\frac{1}{2}(t^2-\frac{r^2}{\beta^2})\right]\right\} \quad (26)$$

Then we substitute $\lambda^*=0.1715$, $\mu^*=0.3007$, $\kappa^*=1.0$, $\rho^*=1.0$, $\rho_f^*=0.4325$, $\alpha=0.779$, $m^*=2.3006$, $M^*=0.3742$, $\gamma(\omega)=0.851e^{1.034i}$; and $\xi_1^*=0.1548$; $\xi_2^*=-1.0076$, $\lambda_1^*=0.991$, and $\lambda_2^*=-0.009$ to Eqns.(25) and (11) respectively, where ξ_1^* and ξ_2^* are the non-dimensional parameters of ξ_1 and ξ_2, respectively; and λ_1^* and λ_2^* are the non-dimensional parameters of λ_1 and λ_2, respectively. The comparisons with both of results are shown in Figures 1through to 6. A good agreement can be discovered easyly.

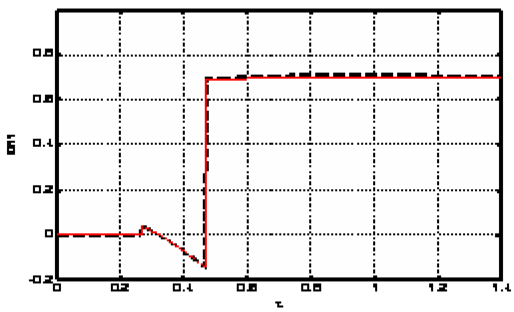


Figure 1. Result of G_{11} compared (the virtual line is from Eqn.(12))

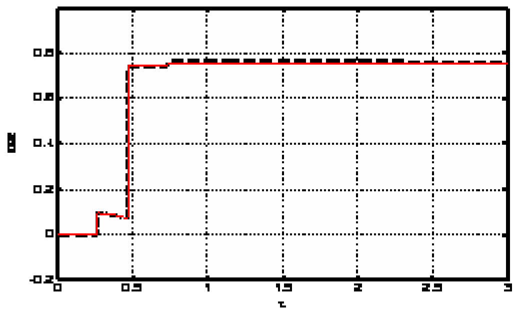


Figure 2. Result of G_{22} compared (the virtual line is from Eqn.(12))

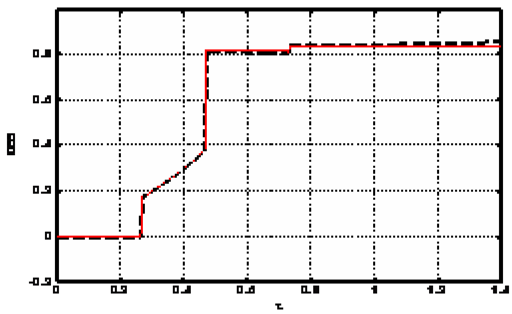


Figure 3. Result of G_{33} compared (the virtual line is from Eqn.(12))

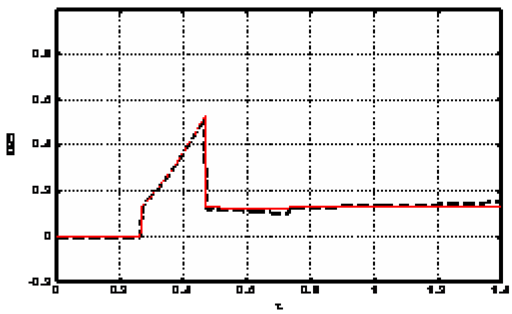
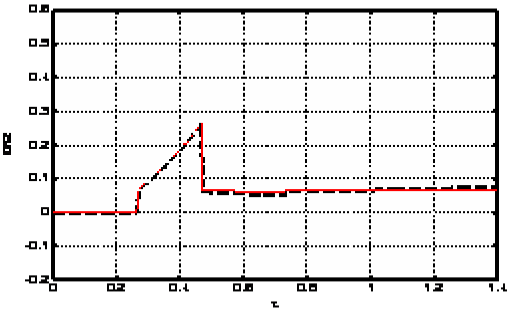


Figure 4. Result of G_{23} compared (the virtual line is from Eqn.(12))



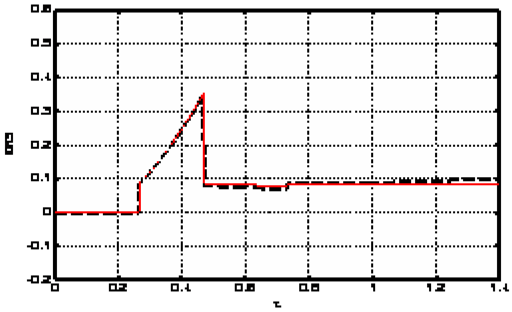
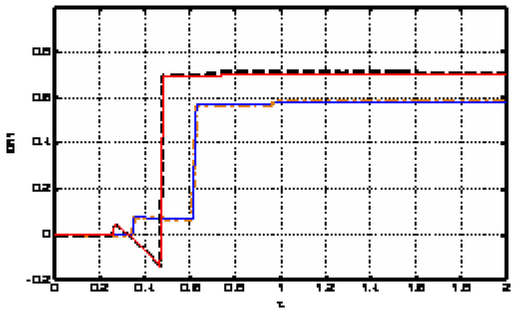


Figure 5. Result of G_{12} compared (the virtual line is from Eqn.(12))

Figure 6. Result of G_{13} compared (the virtual line is from Eqn.(12))

Figures 7 through to 10 are shown the compares with both of results on Eqns.(26) and (12) when the field point (0.1,0.15,0.2) is changed to point (0.2,0.18,0.23). The comparison results (and those obtained for other field points, although not shown here) indicate that calculations based on the present Green's functions are stable.



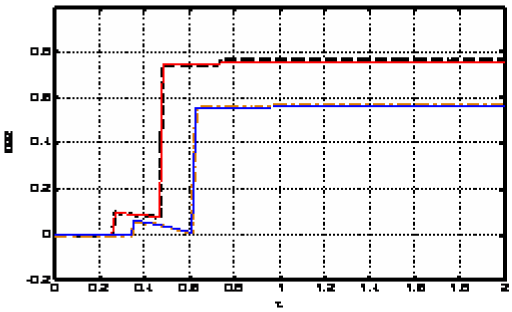
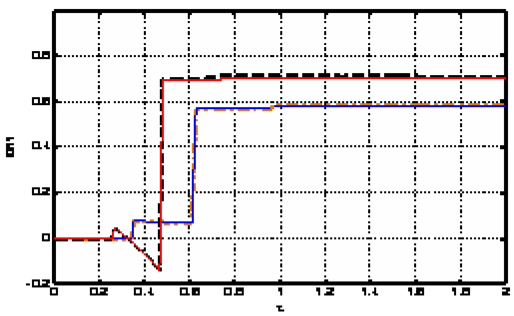


Figure 7. Comparison of G_{11} of arbitrary field point (the virtual line is from Eqn.(12))

Figure 8. Comparison of G_{22} of arbitrary field point (the virtual line is from Eqn.(12))



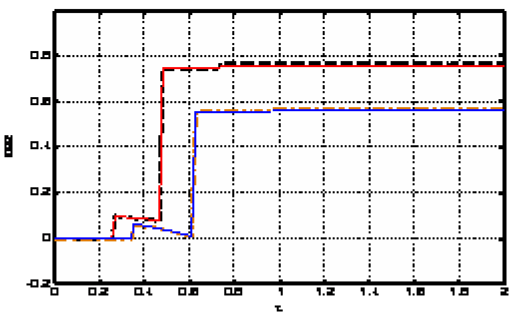


Figure 9. Comparison of G_{11} of arbitrary field point (the virtual line is from Eqn.(12))

Figure 10. Comparison of G_{22} of arbitrary field point (the virtual line is from Eqn.(12))

2. the two-dimensional solution of Green function[21]

The two-dimensional solutions of Green's function are a basic of solving to the dynamic plane strain problem. Manoris indicated that the two-dimensional Green's function g_{ij} of the displacement filed, can be obtained by integrating to the three-dimensional solution G_{ij} along z axis in infinite domain. Integrating Eqn.(26) of aspect z axis in $(-\infty;+\infty)$, we can get then:

$$g_{ln}=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left\{\frac{x_ix_j}{r^3}\left[\lambda_1D_1H(t-\frac{r}{\alpha_1})-\lambda_2D_2H(t-\frac{r}{\alpha_2})-D_3H(t-\frac{r}{\beta})\right]+D_3\frac{\delta_{ij}}{r}H(t-\frac{r}{\beta})\right.$$

$$\left.+3\frac{x_ix_j}{r^5}\left[\alpha_1^2\lambda_1D_1\frac{1}{2}(t^2-\frac{r^2}{\alpha_1^2})-\alpha_2^2\lambda_2D_2\frac{1}{2}(t^2-\frac{r^2}{\alpha_2^2})-\beta^2D_3\frac{1}{2}(t^2-\frac{r^2}{\beta^2})\right]\right\}\mathrm{dz}$$

$$=\frac{1}{2\pi}\left\{\left(D_1\alpha_1^2\left[2t^2-\frac{R^2}{\alpha_1}\right]H\left[t-\frac{R}{\alpha_1}\right]/t_1-D_2\alpha_2^2\left[2t^2-\frac{R^2}{\alpha_2}\right]H\left[t-\frac{R}{\alpha_2}\right]/t_2\right.\right.$$

$$-D_3\beta^2\left[2t^2-\frac{R^2}{\beta^2}\right]H\left[t-\frac{R}{\beta}\right]/t_3)\frac{R_lR_n}{R^4}-\left[D_1\alpha_1^2H(t-\frac{R}{\alpha_1})t_1-D_2\alpha_2^2H(t-\frac{R}{\alpha_2})t_2\right.$$

$$\left.\left.-D_3\beta^2H(t-\frac{R}{\beta})t_3\right]\frac{\delta_{\alpha\beta}}{R^2}+D_3H(t-\frac{R}{\beta})\,\delta_{ln}/t_3\right\}\tag{27}$$

where $t_1=\sqrt{t^2-R^2/\alpha_1{}^2}$, $t_2=\sqrt{t^2-R^2/\alpha_2{}^2}$, $t_3=\sqrt{t^2-R^2/\beta^2}$

Figures 11 through to 13 are the comparison between results in this paper Eqn.(27) and results preexisted Eqn.(28)

$$g_{ij}=\int_{r/\alpha_2}^{t}(P_{11}e^{-b(t-\tau)}+P_{12}e^{-c(t-\tau)}+P_{13})\frac{e^{-\eta_{\alpha_2}\tau}}{\sqrt{\tau^2-r^2/\alpha_2^2}}\cosh(\xi_{\alpha_2}\sqrt{\tau^2-r^2/\alpha_2^2})\mathrm{d}\tau H(t-r/\alpha_2)$$

$$+\int_{r/\alpha_2}^{t}(P_{21}e^{-b(t-\tau)}+P_{22}e^{-c(t-\tau)}+P_{23})e^{-\eta_{\alpha_2}\tau}\sinh(\xi_{\alpha_2}\sqrt{\tau^2-r^2/\alpha_2^2})\mathrm{d}\tau H(t-r/\alpha_2)$$

$$+\int_{r/\alpha_1}^{t}(P_{31}e^{-b(t-\tau)}+P_{32}e^{-c(t-\tau)}+P_{33})\frac{e^{-\eta_{\alpha_1}\tau}}{\sqrt{\tau^2-r^2/\alpha_1^2}}\cosh(\xi_{\alpha_1}\sqrt{\tau^2-r^2/\alpha_1^2})\mathrm{d}\tau H(t-r/\alpha_2)$$

$$+\int_{r/\alpha_1}^{t}(P_{41}e^{-b(t-\tau)}+P_{42}e^{-c(t-\tau)}+P_{43})e^{-\eta_{\alpha_1}\tau}\sinh(\xi_{\alpha_1}\sqrt{\tau^2-r^2/\alpha_1^2})\mathrm{d}\tau H(t-r/\alpha_1)$$

$$+\int_{r/\beta}^{t} P_{51} \frac{e^{-\eta_\beta \tau}}{\sqrt{\tau^2 - r^2/\beta^2}} \cosh(\xi_\beta \sqrt{\tau^2 - r^2/\beta^2}) \mathrm{d}\tau H(t - r/\beta)$$

$$+\int_{r/\beta}^{t} P_{61} e^{-\eta_\beta \tau} \sinh(\xi_\beta \sqrt{\tau^2 - r^2/\beta^2}) \mathrm{d}\tau H(t - r/\beta) \quad (28)$$

where η_{α_1} and $\eta_{\alpha 1}$ are the dissipation factor of fast and slow dilational wave respectively. The meanings of the residual marks on Eqn. (28) can refer to Appendix 2

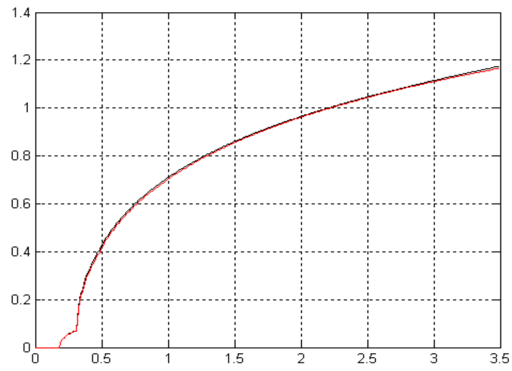


Fig.11 The result of g_{11} compared with Chen

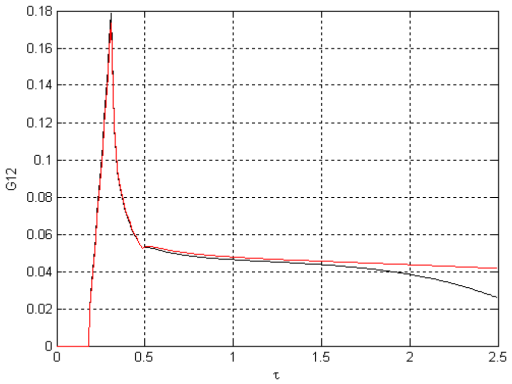


Fig.12 The result of g_{12} compared with Chen

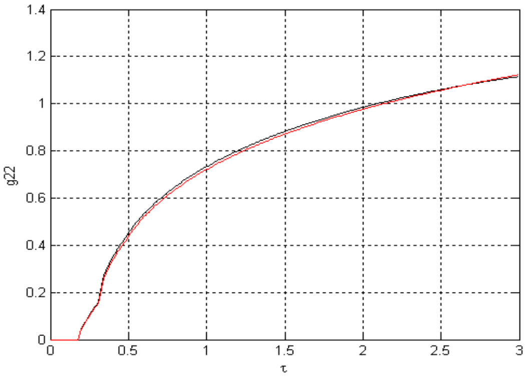


Fig.13 The result of g_{22} compared with Chen

It also can be proved that g_{ln} is stable by the comparison of the results that the coordinate of the field point is changed.

3. Conclusion

3.1. The generalized function δ had been utilized to solve the partial differential equation, but it is infrequent to decompose the δ function to solution the problems. A performing decomposition δ function may be effective and convenient to solve the partial differential equation sometime.

3.2. The three or two dimensional Green's functions provided in this paper can be used as a integral kernel in dynamic BEM.

4. Acknowledgement

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Appendix 1

$$P_{11}=\frac{f_{10}}{c-b}\alpha_2\,,\quad P_{12}=(-\frac{f_{10}}{c-b}+f_{11})\alpha_2\,,\quad P_{13}=f_{12}\alpha_2$$

$$P_{21}=\frac{f_1}{(b-a)(c-a)}-\frac{f_2}{a(b-a)(c-a)}+\frac{f_3}{a^2(b-a)(c-a)}-\frac{f_7}{a^3}$$

$$P_{22}=\frac{f_1}{(a-b)(c-b)}-\frac{f_2}{b(a-b)(c-b)}+\frac{f_3}{b^2(a-b)(c-b)}+\frac{f_4}{c-b}+\frac{f_5}{b(b-c)}$$

$$P_{23}=\frac{f_1}{(a-c)(b-c)}-\frac{f_2}{c(a-c)(b-c)}+\frac{f_3}{c^2(a-c)(b-c)}-\frac{f_4}{c-b}-\frac{f_5}{c(b-c)}+f_6$$

$$P_{24}=\frac{f_2}{abc}-\frac{(ab+bc+ca)f_3}{(abc)^2}+\frac{f_5}{bc}+\frac{f_7}{a^3}+f_8$$

$$P_{25}=\frac{f_3}{abc}-\frac{f_7}{a^2},\quad P_{26}=\frac{f_7}{2a}+\frac{f_9}{2},\quad P_{31}=\frac{g_{10}}{c-b}\alpha_1,\quad P_{32}=(-\frac{g_{10}}{c-b}+g_{11})\alpha_1,\quad P_{33}=g_{12}\alpha_1$$

$$P_{41}=\frac{g_1}{(b-a)(c-a)}-\frac{g_2}{a(b-a)(c-a)}+\frac{g_3}{a^2(b-a)(c-a)}-\frac{g_7}{a^3}$$

$$P_{42}=\frac{g_1}{(a-b)(c-b)}-\frac{g_2}{b(a-b)(c-b)}+\frac{g_3}{b^2(a-b)(c-b)}+\frac{g_4}{c-b}+\frac{g_5}{b(b-c)}$$

$$P_{43}=\frac{g_1}{(a-c)(b-c)}-\frac{g_2}{c(a-c)(b-c)}+\frac{g_3}{c^2(a-c)(b-c)}-\frac{g_4}{c-b}-\frac{g_5}{c(b-c)}+g_6$$

$$P_{44}=\frac{g_2}{abc}-\frac{(ab+bc+ca)g_3}{(abc)^2}+\frac{g_5}{bc}+\frac{g_7}{a^3}+g_8$$

$$P_{45}=\frac{g_3}{abc}-\frac{g_7}{a^2},\quad P_{46}=\frac{g_7}{2a}+\frac{g_9}{2},\quad P_{51}=h_4c_s,\quad P_{61}=-\frac{h_1}{a^3},\quad P_{62}=\frac{h_1}{a^3}+h_2,\quad P_{63}=-\frac{h_1}{a^2}$$

$$P_{64}=\frac{h_1}{2a}+\frac{h_3}{2},\quad a=\frac{1}{\kappa}\frac{\rho}{\rho m-\rho_f^2},\quad b,c=-\frac{1}{2}\left\{-\frac{a_1}{\kappa}\pm\sqrt{\left(\frac{a_1}{\kappa}\right)^2-\frac{4a_2}{\kappa^2}}\right\}$$

$$\eta_\alpha=\frac{1}{2\kappa}\frac{a_7\pm a_4\big/2\sqrt{a_3}}{a_6\pm\sqrt{a_3}},\quad \eta_\beta=\frac{1}{2\kappa m}\frac{\rho_f^2}{m\rho-\rho_f^2},\quad \xi_\beta=\left\{\eta_\beta^2+\frac{1}{\kappa^2}\frac{\rho_f^2}{(m\rho-\rho_f^2)m^2}\right\}^{1/2}$$

$$\xi_\alpha=\left\{\eta_\alpha^2\mp\frac{1}{2\kappa^2}\frac{-a_4^2\big/\left(4a_3^{3/2}\right)+a_5\big/\sqrt{a_3}}{a_6\pm\sqrt{a_3}}\right\}^{1/2}$$, And: $f_1=-\frac{A_{ij}d_1}{\sqrt{a_3}}$, $g_1=\frac{A_{ij}d_1}{\sqrt{a_3}}$, $h_1=-A_{ij}b_1$,

$f_2=-\frac{A_{ij}d_2}{\sqrt{a_3}}$, $g_2=\frac{A_{ij}d_2}{\sqrt{a_3}}$, $h_2=-\frac{C_{ij}}{\mu}+D_{ij}$, $f_3=-\frac{A_{ij}d_3}{\sqrt{a_3}}$, $g_3=\frac{A_{ij}d_3}{\sqrt{a_3}}$, $h_3=-A_{ij}b_2$,

$f_4=\frac{C_{ij}e_2b-C_{ij}e_1}{\sqrt{a_3}}$, $g_4=\frac{-C_{ij}e_2b+C_{ij}e_1}{\sqrt{a_3}}$, $h_4=-\frac{B_{ij}}{\mu}$, $f_5=-\frac{A_{ij}d_4}{\sqrt{a_3}}$, $g_5=\frac{A_{ij}d_4}{\sqrt{a_3}}$,

$f_6=-\frac{C_{ij}e_2}{\sqrt{a_3}}$, $g_6=\frac{C_{ij}e_2}{\sqrt{a_3}}$, $f_7=-A_{ij}d_{62}$; $g_7=A_{ij}d_{61}$, $f_8=-C_{ij}e_{31}$, $g_8=C_{ij}e_{32}$,

$f_9=-A_{ij}d_{52}$, $g_9=A_{ij}d_{51}$, $f_{10}=\frac{-B_{ij}e_1+B_{ij}e_2b}{\sqrt{a_3}}$, $g_{10}=\frac{B_{ij}e_1-B_{ij}e_2b}{\sqrt{a_3}}$, $f_{11}=-\frac{B_{ij}e_2}{\sqrt{a_3}}$,

$g_{11}=\frac{B_{ij}e_2}{\sqrt{a_3}}$, $f_{12}=-B_{ij}e_{31}$, $g_{12}=-B_{ij}e_{32}$;

Where: $d_1=\frac{1}{2}\frac{1}{\rho m-\rho_f^2}\left(\frac{m^2}{M}+\frac{\alpha^2m^2-2\alpha\rho_f m+\rho_f^2}{\lambda+2\mu}\right)$

$$d_2 = \frac{1}{2}\frac{1}{\rho m-\rho_f^2}\left(\frac{2m}{M}+\frac{2\alpha^2 m-2\alpha\rho_f}{\lambda+2\mu}\right)\frac{1}{\kappa}$$

$$d_3 = \frac{1}{2}\frac{1}{\rho m-\rho_f^2}\left(\frac{1}{M}+\frac{\alpha^2}{\lambda+2\mu}\right)\frac{1}{\kappa^2}\,,\quad d_4 = -\frac{1}{2}\frac{1}{\lambda+2\mu}$$

$$d_{51} = \frac{1}{2}\frac{m}{\rho m-\rho_f^2}\,,\quad d_{52} = -\frac{1}{2}\frac{m}{\rho m-\rho_f^2}\,,\quad d_{61} = -\frac{1}{2}\frac{\rho_f^2}{\kappa(\rho m-\rho_f^2)^2}\,,\quad d_{62} = \frac{1}{2}\frac{\rho_f^2}{\kappa(\rho m-\rho_f^2)^2}$$

$$e_1 = \frac{1}{2(\lambda+2\mu)\kappa}\left(\frac{1}{M}-\frac{\alpha^2}{\lambda+2\mu}\right),\quad e_2 = \frac{1}{2(\lambda+2\mu)}\left(\frac{m}{M}+\frac{-\alpha^2 m-\rho+2\alpha\rho_f}{\lambda+2\mu}\right),$$

$$e_{31} = -\frac{1}{2(\lambda+2\mu)}\,,\quad e_{32} = \frac{1}{2(\lambda+2\mu)}\,,\quad b_1 = -\frac{\rho_f^2}{\kappa(\rho m-\rho_f^2)^2}\,,\quad b_2 = \frac{m}{\rho m-\rho_f^2}$$

$$a_1 = \frac{a_4}{2a_3}\,,\quad a_2 = \frac{1}{2}\left(-\frac{1}{4}\frac{a_4^2}{a_3^2}+\frac{a_5}{a_3}\right),\quad a_3 = \left(\frac{\rho+\alpha^2 m-2\alpha\rho_f}{\lambda+2\mu}+\frac{m}{M}\right)^2+\frac{4(\rho_f^2-\rho m)}{M(\lambda+2\mu)}\,,$$

$$a_4 = 2\left[\frac{-\rho+2\alpha^2 m-2\alpha\rho_f}{M(\lambda+2\mu)}+\frac{\alpha^2\left(\rho+\alpha^2 m-2\alpha\rho_f\right)}{(\lambda+2\mu)^2}+\frac{m}{M^2}\right],\quad a_5 = \left(\frac{1}{Q}+\frac{\alpha^2}{\lambda+2\mu}\right)^2$$

$$a_6 = \frac{\rho+\alpha^2 m-2\alpha\rho_f}{\lambda+2\mu}+\frac{m}{M}\,,\qquad a_7 = \frac{1}{M}+\frac{\alpha^2}{\lambda+2\mu}.$$

Appendix 2

$$P_{11} = \frac{f_3}{c-b}\,,\quad P_{31} = \frac{g_3}{c-b}\,,\quad P_{12} = f_2-\frac{f_3}{c-b}\,,\quad P_{32} = g_2-\frac{g_3}{c-b}\,,\quad P_{13} = f_1\,,\quad P_{33} = g_1$$

$$P_{21} = \frac{f_6}{c-b}\frac{\alpha_2^2}{r\xi_d}\,,\quad P_{41} = \frac{g_6}{c-b}\frac{\alpha_1^2}{r\xi_p}\,,\quad P_{22} = (f_5-\frac{f_6}{c-b})\frac{\alpha_2^2}{r\xi_d}\,,\quad P_{42} = (g_5-\frac{g_6}{c-b})\frac{\alpha_1^2}{r\xi_p}$$

$$P_{23} = f_4\frac{\alpha_2^2}{r\xi_d}\,,\quad P_{43} = g_4\frac{\alpha_1^2}{r\xi_p}\,,\quad P_{51} = h_1\,,\quad P_{61} = h_2\frac{\beta^2}{r\xi_s}$$

$$b = -\frac{1}{2}\left\{-a_1/\kappa+\sqrt{(a_1/\kappa)^2-4a_2/\kappa^2}\right\}\,,\qquad c = -\frac{1}{2}\left\{-a_1/\kappa-\sqrt{(a_1/\kappa)^2-4a_2/\kappa^2}\right\}$$

$$\eta_{\alpha_2} = \frac{1}{2\kappa}\frac{a_7+a_4/2\sqrt{a_3}}{a_6+\sqrt{a_3}}\qquad \eta_{\alpha_1} = \frac{1}{2\kappa}\frac{a_7-a_4/2\sqrt{a_3}}{a_6-\sqrt{a_3}}\qquad \eta_\beta = \frac{1}{2\kappa m}\frac{\rho_f^2}{m\rho-\rho_f^2}$$

$$\xi_{\alpha_2} = \left\{\eta_{\alpha_2}^2-\frac{1}{2\kappa^2}\frac{-a_4^2/4a_3^{3/2}+a_5/\sqrt{a_3}}{a_6+\sqrt{a_3}}\right\}^{1/2}$$

$$\xi_{\alpha_1} = \left\{ \eta_{\alpha_1}^2 + \frac{1}{2\kappa^2} \frac{-a_4^2 / 4a_3^{3/2} + a_5 / \sqrt{a_3}}{a_6 - \sqrt{a_3}} \right\}^{1/2}$$

$$\xi_{\beta} = \left\{ \eta_{\beta}^2 + \frac{1}{\kappa^2} \frac{\rho_f^2}{(m\rho - \rho_f^2)m^2} \right\}^{1/2}$$

$$f_1 = -B_{ij} e_{31} \qquad g_1 = B_{ij} e_{32} \qquad h_1 = -\frac{B_{ij}}{\mu} + C_{ij}$$

$$f_2 = -\frac{B_{ij} e_2}{\sqrt{a_3}} \qquad g_2 = \frac{B_{ij} e_2}{\sqrt{a_3}} \qquad h_2 = -\frac{A_{ij}}{\mu}$$

$$f_3 = \frac{-B_{ij} e_1 + B_{ij} e_2 b}{\sqrt{a_3}} \qquad g_3 = \frac{B_{ij} e_1 - B_{ij} e_2 b}{\sqrt{a_3}}$$

$$f_4 = -A_{ij} e_{31} \qquad g_4 = A_{ij} e_{32}$$

$$f_5 = -\frac{A_{ij} e_2}{\sqrt{a_3}} \qquad g_5 = \frac{A_{ij} e_2}{\sqrt{a_3}}$$

$$f_6 = \frac{-A_{ij} e_1 + A_{ij} e_2 b}{\sqrt{a_3}} \qquad g_6 = \frac{A_{ij} e_1 - A_{ij} e_2 b}{\sqrt{a_3}}$$

$$e_1 = \frac{1}{2(\lambda + 2\mu)\kappa} \left(\frac{1}{Q} - \frac{\alpha^2}{\lambda + 2\mu} \right)$$

$$e_2 = \frac{1}{2(\lambda + 2\mu)} \left(\frac{m}{Q} + \frac{-\alpha^2 m - \rho + 2\alpha\rho_f}{\lambda + 2\mu} \right)$$

$$e_{31} = -\frac{1}{2(\lambda + 2\mu)}, \quad e_{31} = \frac{1}{2(\lambda + 2\mu)}$$

$$A_{ij} = \frac{1}{2\pi} \left(\frac{2x_i x_j}{r^3} - \frac{\delta_{ij}}{r} \right) \qquad B_{ij} = \frac{1}{2\pi} \frac{x_i x_j}{r^2} \qquad C_{ij} = \frac{\delta_{ij}}{2\pi\mu} \qquad r^2 = x_i x_i \qquad i, j = 1, 2$$

$$a_1 = \frac{a_4}{2a_3}$$

$$a_2 = \frac{1}{2} \left(-\frac{1}{4} \frac{a_4^2}{a_3^2} + \frac{a_5}{a_3} \right)$$

$$a_3 = \left(\frac{\rho + \alpha^2 m - 2\alpha\rho_f}{\lambda + 2\mu} + \frac{m}{M} \right)^2 + \frac{4(\rho_f^2 - \rho m)}{M\left(\lambda + 2\mu\right)}$$

$$a_4 = 2\left[\frac{-\rho + 2\alpha^2 m - 2\alpha\rho_f}{M(\lambda + 2\mu)} + \frac{\alpha^2\left(\rho + \alpha^2 m - 2\alpha\rho_f\right)}{(\lambda + 2\mu)^2} + \frac{m}{M^2} \right]$$

$$a_5 = \left(\frac{1}{M} + \frac{\alpha^2}{\lambda + 2\mu} \right)^2$$

$$a_6 = \frac{\rho + \alpha^2 m - 2\alpha\rho_f}{\lambda + 2\mu} + \frac{m}{M}, \quad a_7 = \frac{1}{M} + \frac{\alpha^2}{\lambda + 2\mu}$$