Miscellaneous open problems in the Regular Boundary Collocation approach

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Basic formulations on a simple example

Laplace equation with Dirichlet conditions

$$\nabla^2 \varphi(x, y) = 0 \qquad (x, y) \in \Omega \qquad (1)$$

$$\varphi(x, y) = f(x, y) \qquad (x, y) \in \Gamma \qquad (2)$$

$$f(x, y) \text{ regular on } \Gamma$$

(1)–(2): Boundary Value Problem

$$\varphi(x, y) \approx \hat{\varphi}(x, y) = \sum_k a_k \Phi_k(x, y) \equiv \mathbf{\Phi a} \qquad \mathbf{\Phi} \equiv \{\Phi_1, \Phi_2, \ldots . \Phi_K\}, \quad \mathbf{a}^T \equiv \{a_1, a_2, \ldots . a_K\} \quad (3)$$

Φ_k - *Trefftz trial functions – fulfilling homogeneous equation inside* Ω *including* Γ

Trefftz-type trial functions

$\Phi_k^H(x,y)$ - Herrera functions: complete sets of regular solutions of Eq.(1) (for open areas Ω singularity in central point $(0.0, 0.0)$)

$\Phi_k^K(x,y)$ - Kupradze functions: fundamental solutions with singularities outside Ω

$\Phi_k^O(x,y)$ - other Trefftz functions: e.g. fundamental solutions expanded into Fourier series or in fuzzy form (singularity extended to finite area)

Remark: functions $\Phi_k(x,y)$ singular on Γ (see BEM) in the Trefftz approach are excluded.

Open problems:
mixed-type Trefftz functions, location of singularities of fundamental solutions

Basic formulations on simple example

$$I = \int_{\Gamma} \left(\hat{\varphi} - f \right)^2 d\Gamma = \int_{\Gamma} \left[\sum_{k}^{K} a_k \Phi_k - f \right]^2 d\Gamma = \min \qquad (4)$$

$$\frac{\partial I}{\partial a_j} = 0 \qquad \Rightarrow \qquad \int_{\Gamma} \Phi_j \left(\hat{\varphi} - f \right) d\Gamma = 0 \qquad (5)$$

$$j = 1,2,3 \ldots K$$

$$\sum_{k}^{K} a_k \int_{\Gamma} \Phi_j \Phi_k \, d\Gamma = \int_{\Gamma} \Phi_j f \, d\Gamma \qquad (6)$$

$$j = 1,2,3 \ldots K$$

simplest orthogonalization (other weighting functions possible)

Numerical integration

$$\int_{\Gamma} h(x,y)\, d\Gamma \;\Rightarrow\; \sum_{L} \int_{s_{1L}}^{s_{2L}} \tilde{h}(s)\, ds \;\Rightarrow\; \sum_{L}\sum_{\mu} \alpha_{\mu L}\, h(x_{\mu L}, y_{\mu L}) \overset{notation}{=} \sum_{\mu}^{\Gamma} [h(x,y)] \qquad (7)$$

$$(x_{\mu L}, y_{\mu L}) \in \Gamma$$

$$\alpha_{\mu L} = \frac{s_{2L} - s_{1L}}{2}\, \tilde{\alpha}_{\mu}$$

L - *segments of* Γ

$\tilde{\alpha}_{\mu}$ - *pure weights of integration*

(x_{μ}, y_{μ}) - *control points of integration on* Γ

Numerical integral formulation

$$\int_{\Gamma} (\hat{\varphi} - f)^2 d\Gamma = \min \quad \overset{notation}{\Rightarrow} \quad \sum_{\mu}^{\Gamma} (\hat{\varphi} - f)^2 = \min \qquad (8)$$

$$\sum_{k}^{K} a_k \sum_{\mu}^{M} \alpha_\mu \Phi_{k\mu} \Phi_{j\mu} = \sum_{\mu}^{M} \alpha_\mu \Phi_{j\mu} f_\mu \qquad j = 1, 2, 3\ldots \qquad (9)$$

$$\Phi_{k\mu} \equiv \Phi_k (x_\mu, y_\mu)$$

(x_μ, y_μ) *- control points of integration on* Γ

α_μ *- weights of integration*

More general: boundary collocation

$$\sum_{\mu}^{\Gamma} (\hat{\varphi} - f)^2 = \min \qquad (10)$$

$$\sum_{k}^{K} a_k \sum_{\nu}^{M} \beta_\nu \Phi_{k\nu} \Phi_{j\nu} = \sum_{\nu}^{M} \beta_\nu \Phi_{j\nu} f_\nu \qquad j = 1, 2\ldots K \qquad \Phi_{k\nu} \equiv \Phi_k (x_\nu, y_\nu) \qquad (11)$$

(x_ν, y_ν) *- control points of collocation on* Γ

β_ν *- weights of collocation*

Numerical boundary formulation

Open problem: choice of β_ν and (x_ν, y_ν)

$\beta_\nu = \alpha_\nu$

(x_ν, y_ν) - *control points of integration on* Γ

} *Numerical integral – specific case of boundary collocation*

Examples of control points of collocation along Γ :

- equdistant points,
- Gaussian points,
- Lobatto points.

Conclusion: proposed general name

Trefftz Method

MFS

} Regular Boundary Collocation Method

Specific case of boundary collocation: $K = M$

M - *number of control points*

Formal integral notation:

$$\int_{\Gamma} (\hat{\varphi} - f)\, \delta(\mathbf{x} - \mathbf{x}_{\mu})\, d\Gamma = 0 \qquad (12)$$

selecting property of $\delta(\mathbf{x} - \mathbf{x}_{\mu})$

$\mu = 1,2,3 \ldots \mathrm{M}$

$$\beta_{\mu} \sum_{k}^{K} a_k \Phi_{k\mu} = \beta_{\mu} f_{\mu} \qquad (x_{\mu}, y_{\mu}) \in \Gamma \qquad (13)$$

$\mu = 1,2,3 \ldots M$

Interpolation;

β_{μ} *have no influence on results*

Matrix notation for $K = M$:

$$\mathbf{B}\,\mathbf{a} = \mathbf{f}$$

$\mathbf{B}$ - *square matrix*

Case: $M > K$

$\mathbf{B}^T \mathbf{B}\,\mathbf{a} = \mathbf{B}^T \mathbf{f}$ *least square*

Integral case:

$$\int_{\Gamma} \Phi_k \left(\hat{\varphi} - f \right) d\Gamma = 0 \qquad (14)$$

$$k = 1,2,3 \ldots K$$

M - *number of integral control points*

$M = K$ - *interpolation; integral weights do not influence results*

$M > K$ - *overdetermined, least square*

$M < K$ - *approximation underdetermined*

General form of collocation

Direct weighting

$$\sum_{\nu}^{\Gamma} \Phi_k \left(\hat{\varphi} - f \right) = 0 \qquad (15)$$

$$k = 1,2,3 \ldots K$$

Opposite weighting

$$\sum_{\nu}^{\Gamma} \Phi_k^{\bullet} \left(\hat{\varphi} - f \right) = 0 \qquad (16)$$

$$k = 1,2,3 \ldots K$$

$$\Phi_k^{\bullet} = \frac{\partial \Phi_k}{\partial n}$$

n - *outward normal to* Γ

Attention: $\Phi_k^H \Big|_{k=1} = const$ $\rightarrow$ necessary additional equation

Open problem: other types of weighting functions

Mixed Dirichlet-Neuman boundary conditions

$$\sum_{\mu}^{\Gamma_1}\left(\hat{\varphi}-f\right)^2+W^2\sum_{\mu}^{\Gamma_2}\left(\frac{\partial\hat{\varphi}}{\partial n}-g\right)^2=\min \qquad (17)$$

$$\sum_{\mu}^{\Gamma_1}\Phi_k\left(\hat{\varphi}-f\right)+W^2\sum_{\mu}^{\Gamma_2}\Phi_k^{\bullet}\left(\frac{\partial\hat{\varphi}}{\partial n}-g\right)^2=0 \qquad (18)$$

$$k=1,2,3\ldots K$$

Open problem: choice of W

Mixed Dirichlet-Neuman boundary conditions

$$\sum_{\mu}^{\Gamma_1} \Phi_k \left(\sum_i a_i \, \Phi_i - f \right) + W^2 \sum_{\mu}^{\Gamma_2} \Phi_k^{\bullet} \left(\sum_i a_i \, \Phi_i^{\bullet} - g \right) = 0 \qquad (19)$$

$$k = 1,2,3,\ldots K$$

$$\Phi_k^{\bullet} = \frac{\partial \Phi_k}{\partial n}$$

$$\sum_i a_i \left[\underbrace{\sum_{\mu}^{\Gamma_1} \Phi_k \Phi_i}_{\text{terms 1}} + W^2 \underbrace{\sum_{\mu}^{\Gamma_2} \Phi_k^{\bullet} \Phi_i^{\bullet}}_{\text{terms 2}} \right] = \ldots \qquad k = 1,2,3\ldots K$$

If terms 1 >> terms 2, condition on Γ_2 will not be fulfilled

Opposite weighting:

$$\sum_{\mu}^{\Gamma_1} \Phi_k^{\bullet}(\hat{\varphi} - f) - \sum_{\mu}^{\Gamma_2} \Phi_k \left(\frac{\partial \hat{\varphi}}{\partial n} - g \right) = 0 \qquad (20)$$

$$k = 1, 2, \ldots K$$

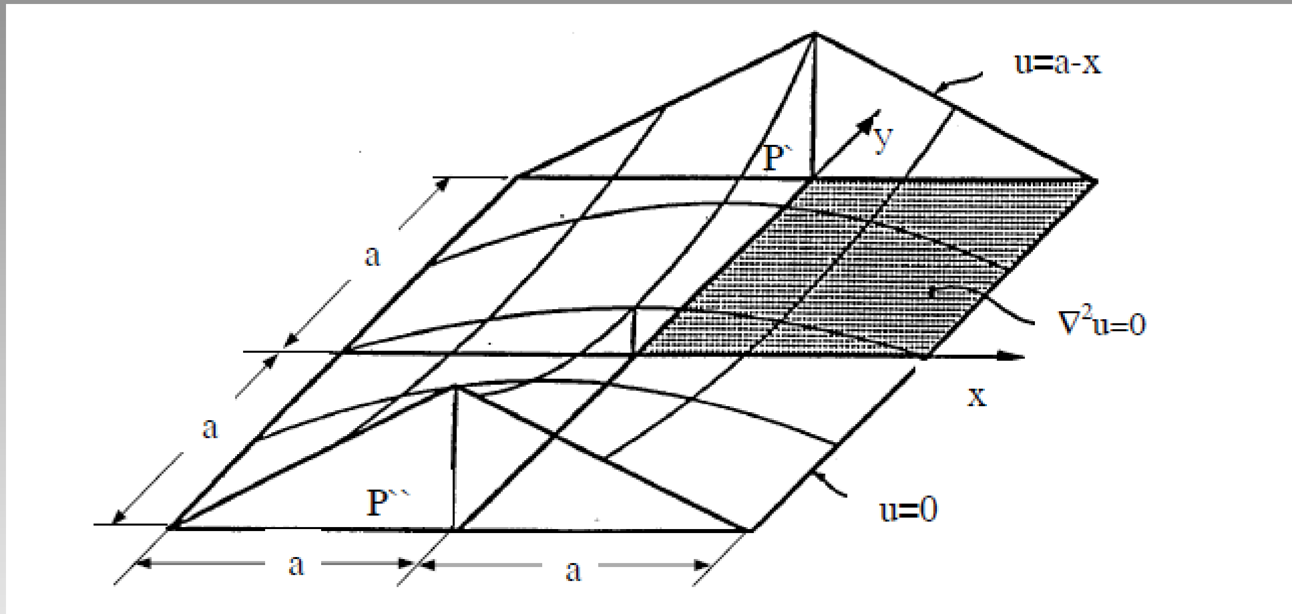
- weight not necessary
- symmetric stiffness matrix (approximately)
- possibility of negative definite matrices

$$\sum_{\mu}^{\Gamma_1} \Phi_k^{\bullet}(\hat{\varphi} - f) + \sum_{\mu}^{\Gamma_2} \Phi_k \left(\frac{\partial \hat{\varphi}}{\partial n} - g \right) = 0 \qquad (21)$$

$$k = 1, 2, \ldots K$$

P. Ladeveze, Ch.Hochard
- nonsymmetric matrices

Numerical example

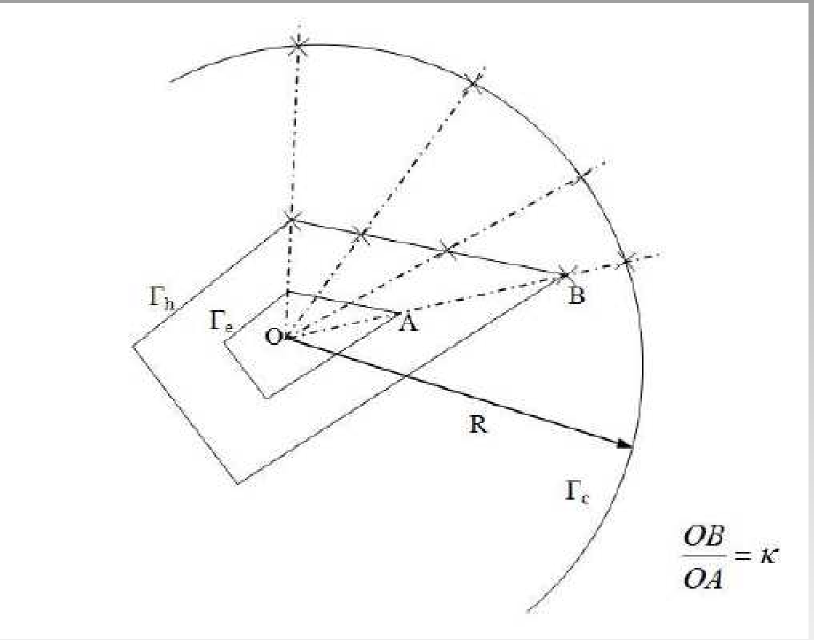


Singular membrane problem.

$$u(x,y) = \sum_{n=0}^{N} \frac{8a\cos[(2n+1)\pi x/(2a)]\cosh[(2n+1)\pi y/(2a)]}{(2n+1)^2\pi^2\cosh[(2n+1)\pi/2]} \qquad (22)$$

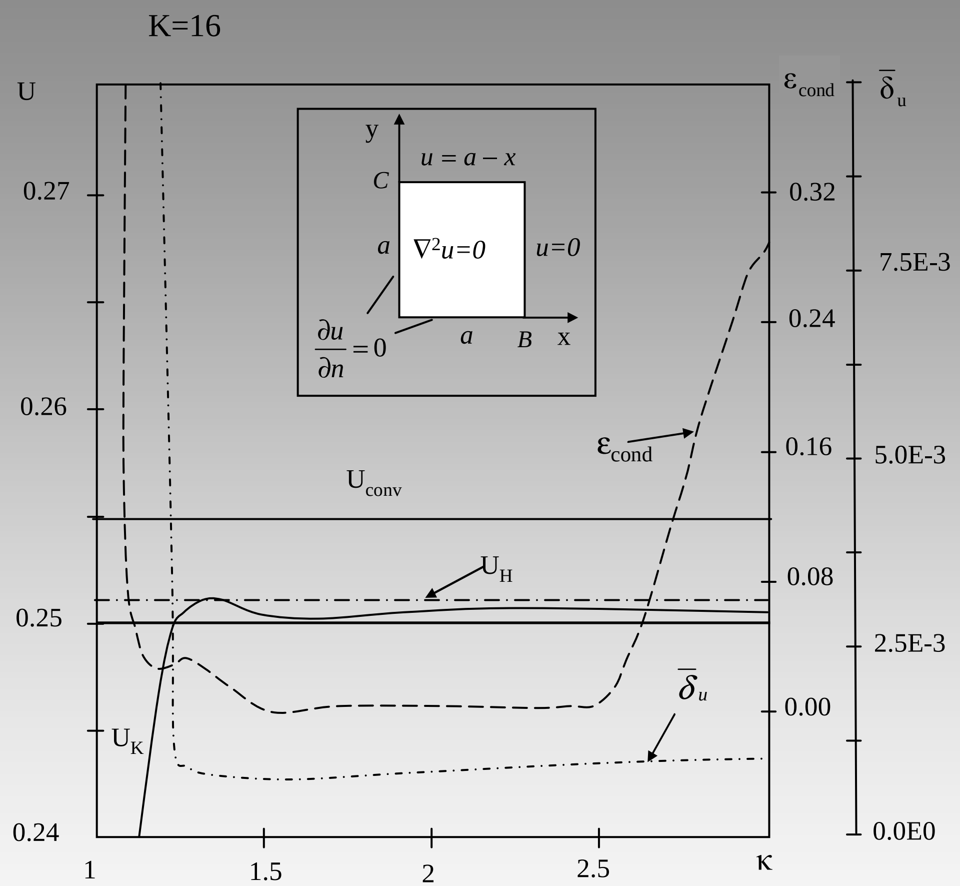
$$N \to \infty \qquad (in\ the\ example\ N = 200)$$

Numerical example: singular membrane problem.



Position of singularities of fundamental solutions (Kupradze functions)

Numerical example: singular membrane problem.



$$\delta_u = \frac{u^{K,H} - u^{EX}}{u_c^{EX}}$$

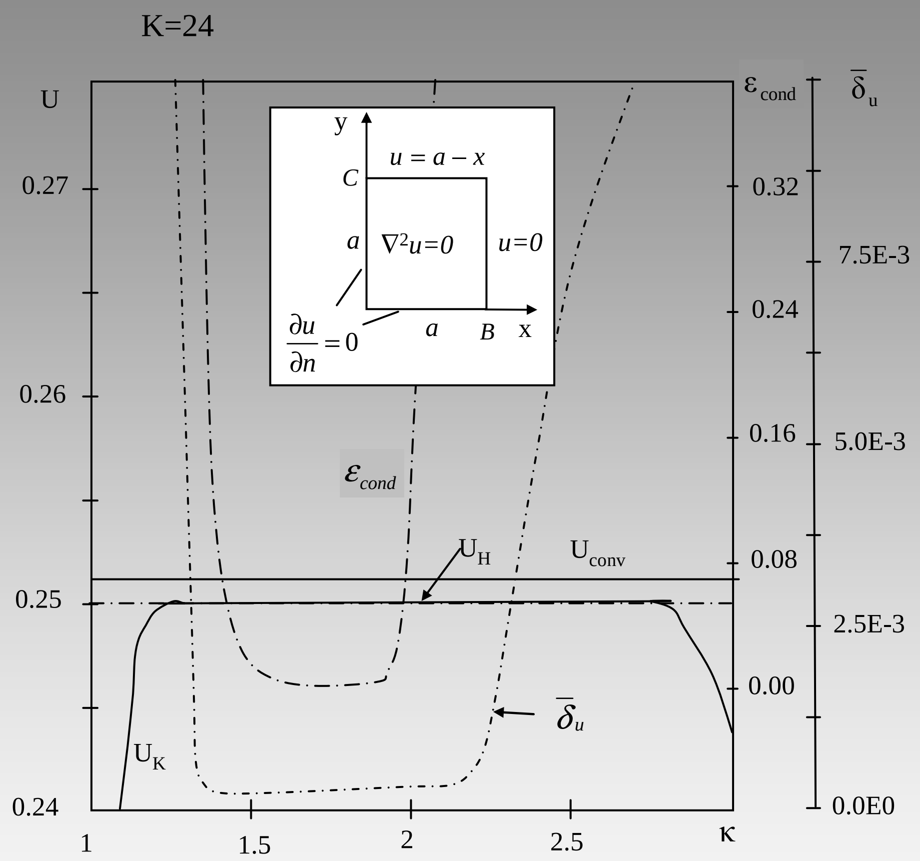
$$\bar{\delta}_u = \frac{\sum_{i,j=1}^{N} \left| u_{i,j}^{K,H} - u_{i,j}^{EX} \right|}{N^2 u_c^{EX}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \left| \delta_u \right|_{ij}$$

$$U^{EX} = \frac{1}{2} \int_{\Omega} \left[\left(\frac{\partial u^{EX}}{\partial x} \right)^2 + \left(\frac{\partial u^{EX}}{\partial y} \right)^2 \right] d\Omega = 0.25a^2$$

$$\varepsilon_{\text{cond}} = \max_i \left| (\mathbf{H}^{-1}\mathbf{H})_{ii} - 1 \right|$$

Range of acceptable distance from boundary Γ_e of singularities of the Kupradze functions: homothetic contour Γ_h. Generalized energy calculated with help of a single conventional p-element (U_{conv}), HT-H approach (U_H), and HT-K formulation (U_K). $a = 1$, $U^{EX} = 0.25$

Numerical example: singular membrane problem.



$$\delta_u = \frac{u^{K,H} - u^{EX}}{u_c^{EX}}$$

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$$U^{EX} = \frac{1}{2} \int_{\Omega} \left[\left(\frac{\partial u^{EX}}{\partial x} \right)^2 + \left(\frac{\partial u^{EX}}{\partial y} \right)^2 \right] d\Omega = 0.25 a^2$$

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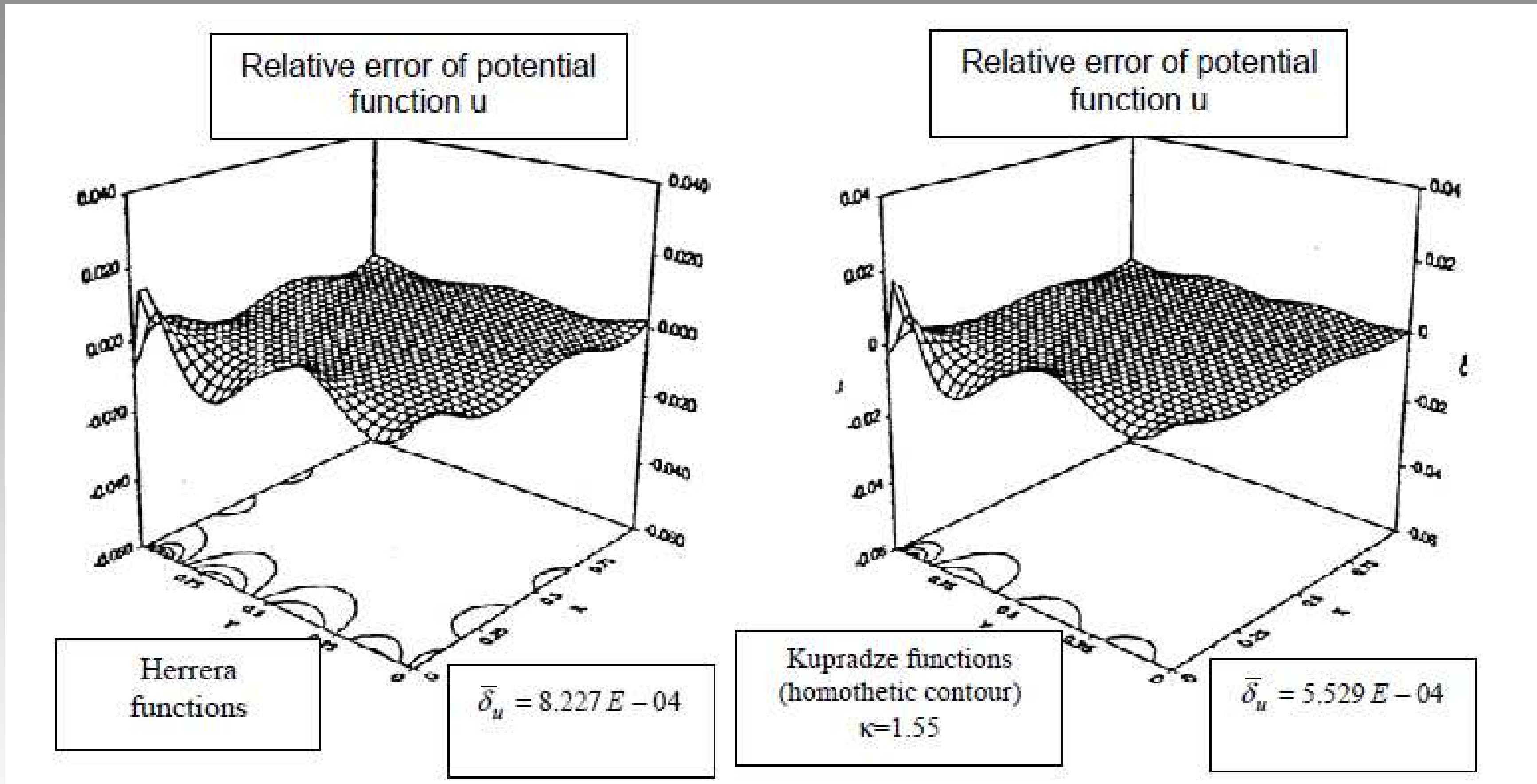
Range of acceptable distance from boundary Γ_ε of singularities of the Kupradze functions: homothetic contour Γ_λ. Generalized energy calculated with help of a single conventional p-element (U_{conv}), HT-H approach (U_H), and HT-K formulation (U_K). $a = 1$, $U^{EX} = 0.25$, $u_c^{EX} = 1.0$

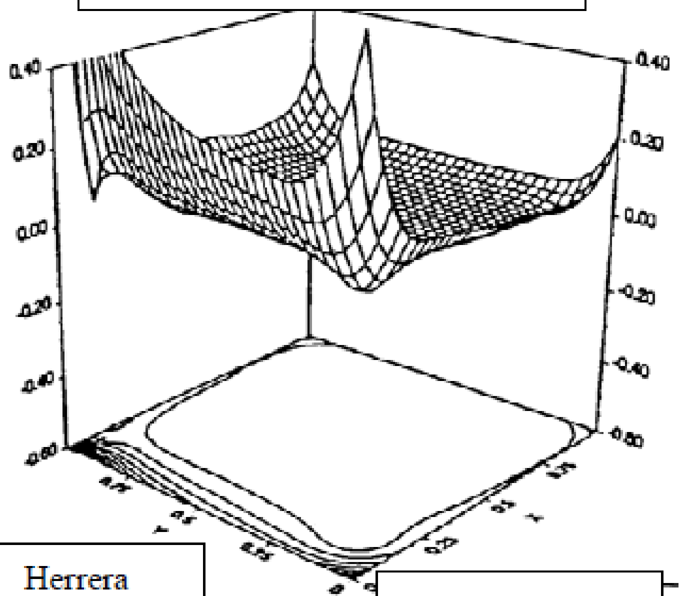
Numerical example: singular membrane problem.



$$\overline{\delta}_u = \frac{\sum_{i,j=1}^{N} \left| u_{i,j}^{K,H} - u_{i,j}^{EX} \right|}{N^2 u_c^{EX}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \left| \delta_u \right|_{ij}$$

Numerical example: singular membrane problem.

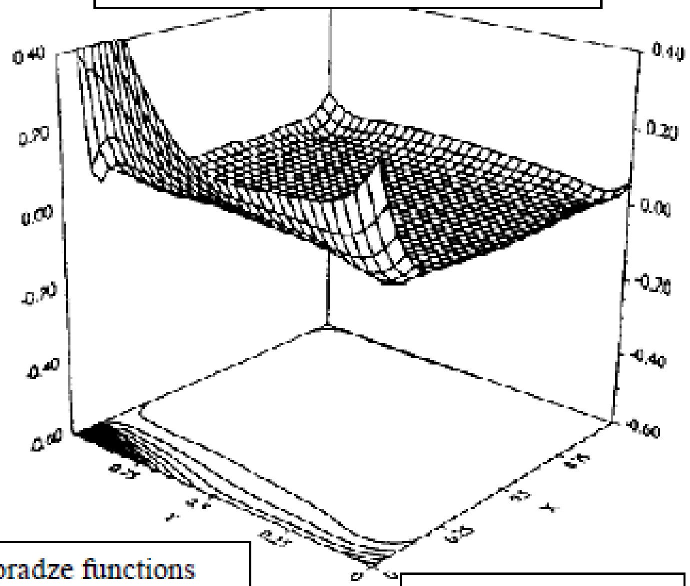
Distribution of energy density error for potential problem



Herrera functions

$\bar{\gamma}_u = 6.416E-02$

Distribution of energy density error for potential problem



Kupradze functions (homothetic contour) κ=1.55

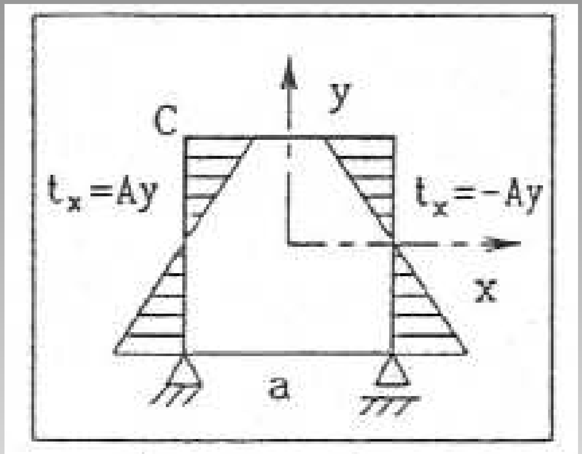
$\bar{\gamma}_u = 4.617E-02$

$$\gamma_U = \left\{ \frac{\left[\frac{\partial}{\partial x}\left(u^{K,H} - u^{EX}\right)\right]^2 + \left[\frac{\partial}{\partial y}\left(u^{K,H} - u^{EX}\right)\right]^2}{2\rho^{EX}} \right\}^{1/2}$$

$$\rho^{EX} = \frac{U^{EX}}{a^2} = 0.25$$

$$\bar{\gamma}_U = \frac{1}{N^2}\sum_{i,j=1}^{N} \gamma_{Uij}$$

Numerical example: 2D elasticity



$$u^{EX}(x,y) = \frac{A}{E}(1 - xy), \quad \upsilon^{EX}(x,y) = \frac{A}{2E}\left(x^2 + \nu y^2 - \nu - 1\right) \quad .$$

$$\delta_u = \left\{ \frac{\left(u^{K,H} - u^{EX}\right)^2 + \left(\upsilon^{K,H} - \upsilon^{EX}\right)^2}{\left(u_C^{EX}\right)^2 + \left(\upsilon_C^{EX}\right)^2} \right\}^{1/2}$$

$$\bar{\delta}_u = \frac{1}{N^2} \sum_{i,j=1}^{N} \delta_{u\,ij}$$

Energy density error $\rightarrow$

$$\gamma_U = \left\{ \frac{1}{E\rho^{EX}} \left[\frac{1}{2}\left(\sigma_x^{K,H} - \sigma_x^{EX}\right)^2 + \frac{1}{2}\left(\sigma_y^{K,H} - \sigma_y^{EX}\right)^2 - \nu\left(\sigma_x^{K,H} - \sigma_x^{EX}\right)\left(\sigma_y^{K,H} - \sigma_y^{EX}\right) + \right.\right.$$

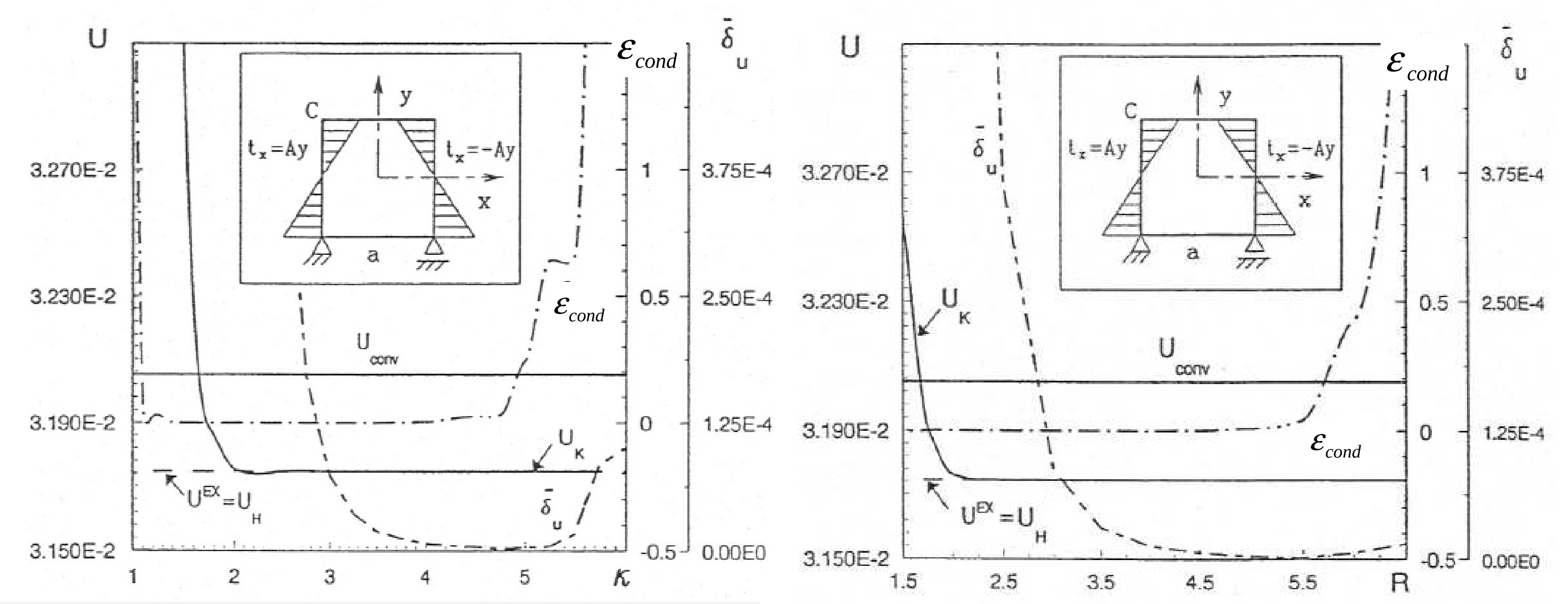
$$\left.\left. + (1+\nu)\left(\tau_{xy}^{K,H} - \tau_{xy}^{EX}\right)^2 \right] \right\}^{1/2}, \qquad \rho^{EX} = \frac{U^{EX}}{a^2}$$

$$\bar{\gamma}_U = \frac{1}{N^2} \sum_{i,j=1}^{N} \gamma_{U\,ij}$$

E - Young modulus

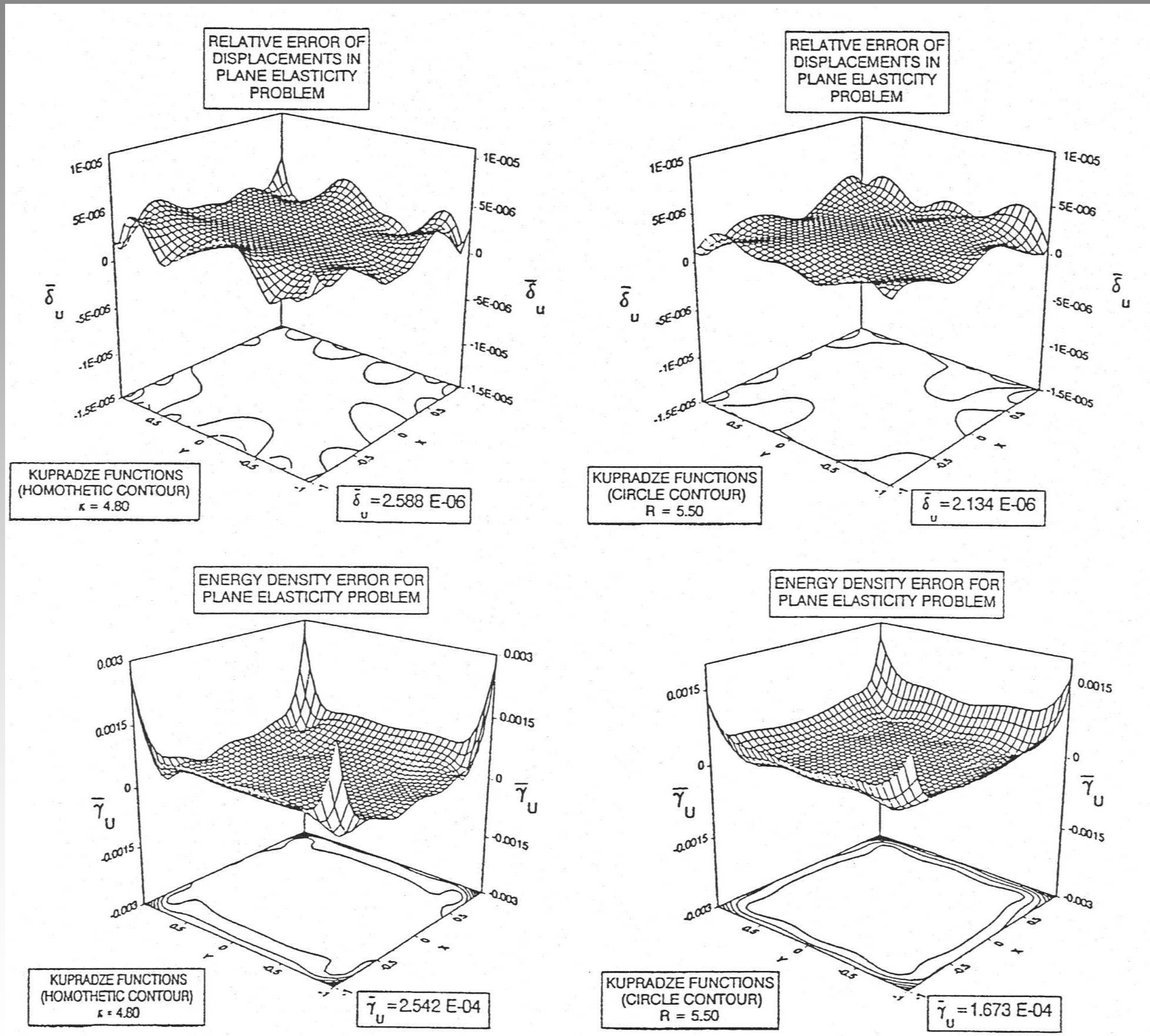
ν - Poisson ratio

Numerical example: 2D elasticity

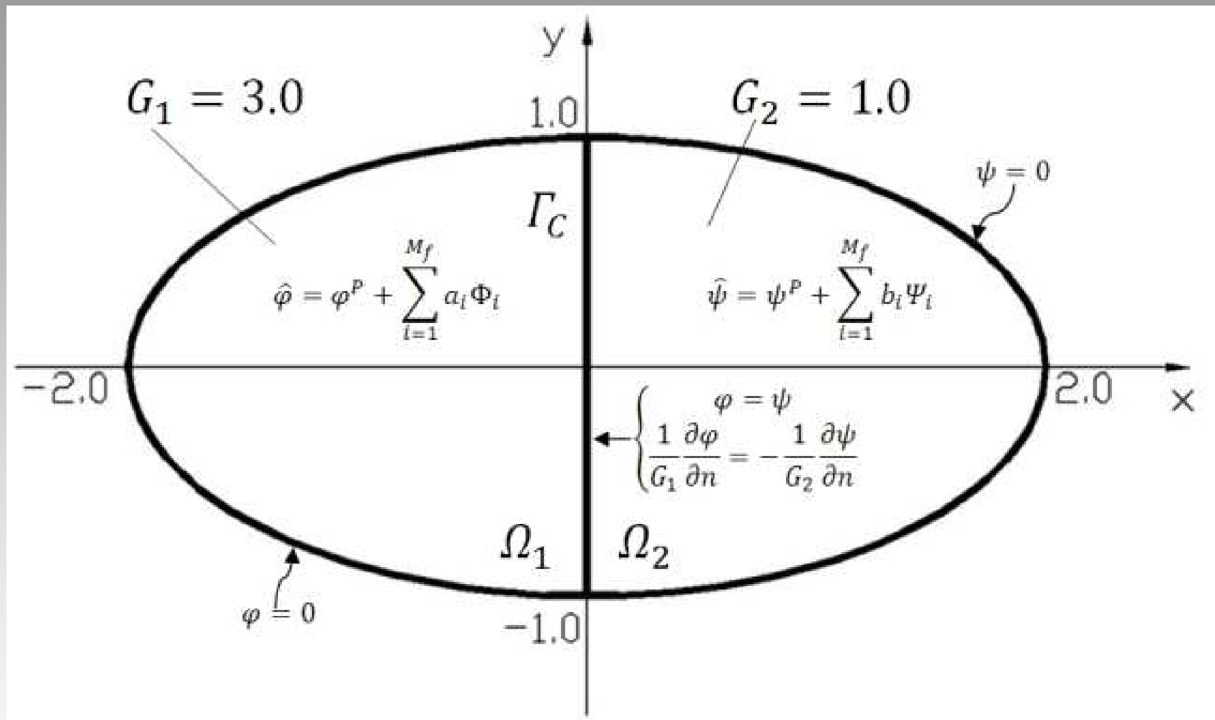


Range of acceptable distance from the element boundary of singularities of Kupradze functions in HT-K solutions. Kupradze functions with singularities situated (a) on a homothetic contour and (b) on a circle: $K = 12$, $a = 2$, $A = 100$. Generalized energy calculated with help of a single conventional p-element (U_{conv}), HT-H approach (U_H), and HT-K formulation (U_K). $U^{EX} = 0.031746$.

Numerical example: 2D elasticity



Coupling of two subregions
Torsion of a bar made of different materials



Cross-section of an elliptic bar made of two different materials – example

Coupling of two subregions
Torsion of a bar made of different materials

$$\nabla^T \frac{1}{G_1} \nabla \phi = -2\beta \qquad (x,y) \in \Omega_1$$

$$\nabla^T \frac{1}{G_2} \nabla \psi = -2\beta \qquad (x,y) \in \Omega_2$$

$$\phi = 0 \qquad (x,y) \in \Gamma_1$$

$$\psi = 0 \qquad (x,y) \in \Gamma_2$$

$$\left.\begin{aligned} &\phi = \psi \\ &\frac{1}{G_1}\frac{\partial \phi}{\partial n} = -\frac{1}{G_2}\frac{\partial \psi}{\partial n} \end{aligned}\right\} \qquad (x,y) \in \Gamma_c$$

β - *angle of twist per unit bar length*

φ, ψ - *Prandtl functions*

G_1 , G_2 - *Kirhchoff moduli*

Coupling of two subregions
Torsion of a bar made of different materials

$$\int_{\Gamma_1} W_j^{(1)} \hat{\phi}\, d\Gamma_1 + \int_{\Gamma_c} \left[W_j^{(2)} \left(\hat{\phi} - \hat{\psi} \right) + W_j^{(3)} \left(\frac{1}{G_1} \hat{\phi}^{\bullet} + \frac{1}{G_2} \hat{\psi}^{\bullet} \right) \right] d\Gamma_c = 0$$

$$j = 1, 2, \ldots K$$

$$\int_{\Gamma_c} \left[W_j^{(4)} \left(\frac{1}{G_2} \hat{\psi}^{\bullet} + \frac{1}{G_1} \hat{\phi}^{\bullet} \right) + W_j^{(5)} \left(\hat{\psi} - \hat{\phi} \right) \right] d\Gamma_c + \int_{\Gamma_2} W_j^{(6)} \hat{\psi}\, d\Gamma_2 = 0$$

$$\hat{\phi} = \sum_k a_k^1 \Phi_k + \phi^p$$

$$\hat{\psi} = \sum_k a_k^2 \Psi_k + \psi^p$$

$$\hat{\phi}^{\bullet} \equiv \frac{\partial \hat{\phi}}{\partial n} = \sum_k a_k^1 \Phi_k^{\bullet} + \phi^{\bullet p}$$

$$\hat{\psi}^{\bullet} \equiv \frac{\partial \hat{\psi}}{\partial n} = \sum_k a_k^2 \Psi_k^{\bullet} + \psi^{\bullet p}$$

$\phi^p,\ \psi^p$ *– particular solutions of nonhomogeneous equations*

Coupling of two subregions
Torsion of a bar made of different materials

$$W_j^{(1)} = W_j^{(2)} = \Phi_j \qquad W_j^{(3)} = \frac{1}{G_1}\Phi_j^{\bullet}$$

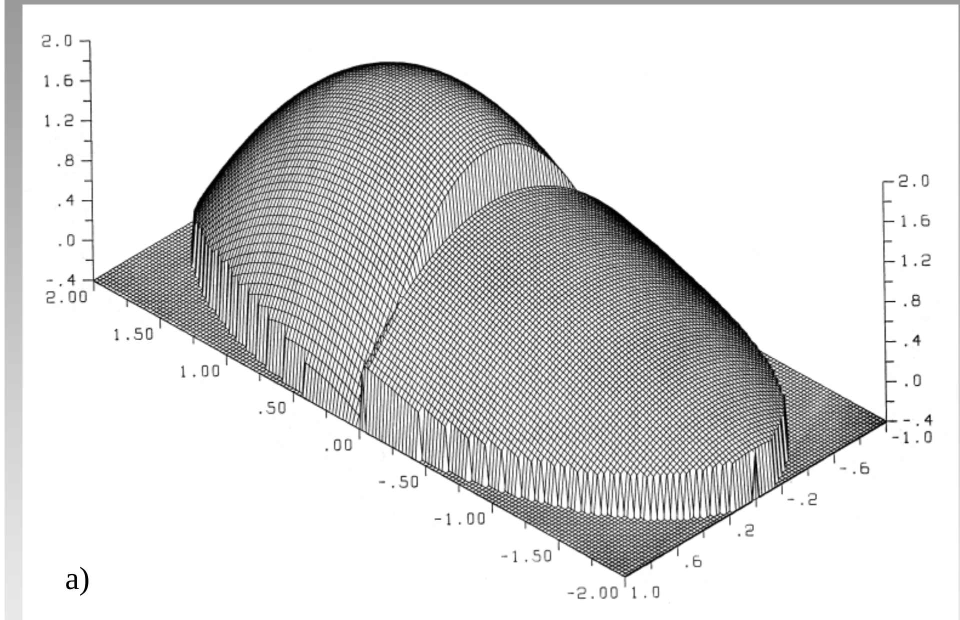
$$W_j^{(5)} = W_j^{(6)} = \Psi_j \qquad W_j^{(4)} = \frac{1}{G_2}\Psi_j^{\bullet}$$

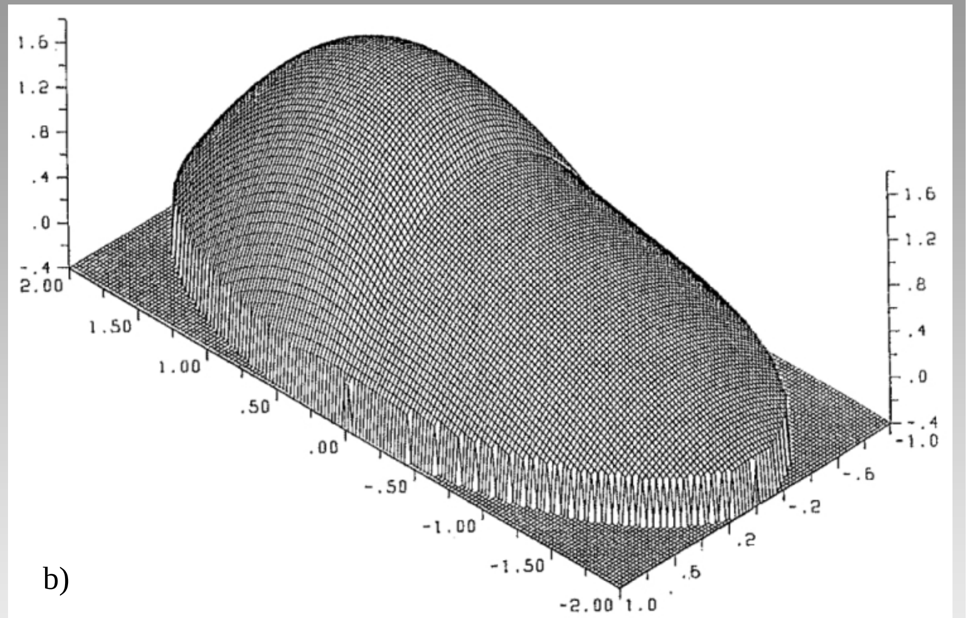
K a = f

$$K_{jk} = \left\| \begin{array}{c|c} \int\limits_{\Gamma_{1+c}} \Phi_j \Phi_k \, d\Gamma_{1+c} + \frac{1}{G_1^2} \int\limits_{\Gamma_c} \Phi_j^{\bullet} \Phi_k^{\bullet} \, d\Gamma_c & -\int\limits_{\Gamma_c} \left(\Phi_j \Psi_k - \frac{1}{G_1 G_2} \Phi_j^{\bullet} \Psi_k^{\bullet} \right) d\Gamma_c \\ \hline -\int\limits_{\Gamma_c} \left(\Psi_j \Phi_k - \frac{1}{G_1 G_2} \Psi_j^{\bullet} \Phi_k^{\bullet} \right) d\Gamma_c & \int\limits_{\Gamma_{2+c}} \Psi_j \Psi_k \, d\Gamma_{2+c} + \frac{1}{G_2^2} \int\limits_{\Gamma_c} \Psi_j^{\bullet} \Psi_k^{\bullet} \, d\Gamma_c \end{array} \right\|$$

$$j, k = 1, 2, \ldots K$$

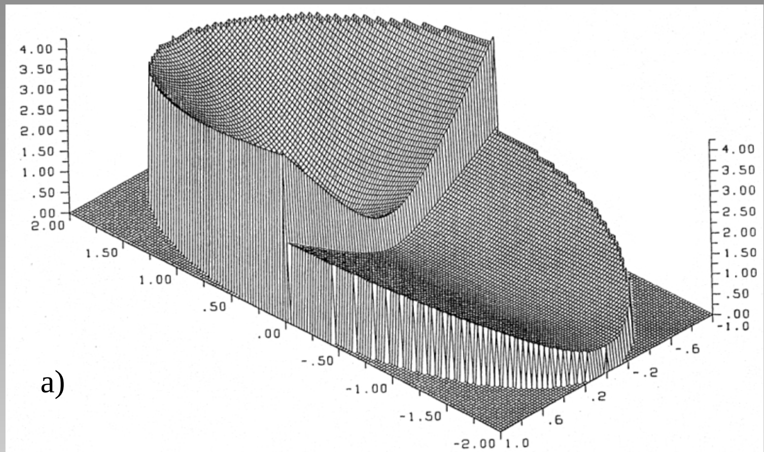
Coupling of two subregions
Torsion of a bar made of different materials

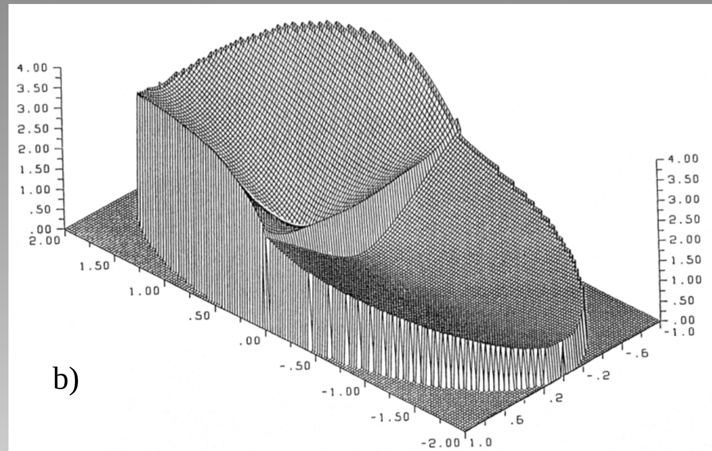


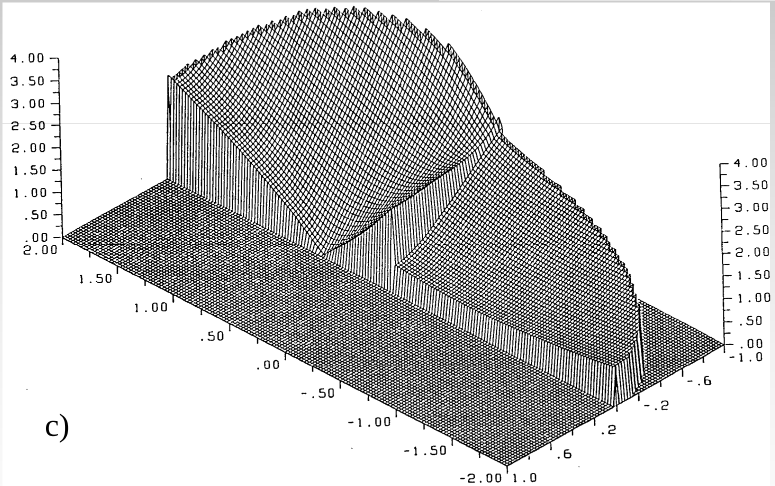


Direct coupling of the two half-elliptic regions – stress function, $G1$=3.0, $G2$=1.0, Herrera functions, a) K= 3; b) K = 9;

Coupling of two subregions
Torsion of a bar made of different materials







$$\tau = \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial x} \right)^2 \right]^{1/2}$$

Direct coupling of the two half-elliptic regions – resulting stresses τ , $G1 = 3.0$, $G2 = 1.0$:
Herrera functions, a) $K = 3$; b) $K = 9$; c) $K = 9$ – cross-section

Coupling of many subregions (elements) – J. Jirousek 1994

Example: Laplace equation

$$J(\mathbf{a}) = \int_{\Gamma_\varphi} (\hat{\varphi} - \overline{\varphi})^2 \, d\Gamma + W_{qn}^2 \int_{\Gamma_{\varphi n}} \left[(\hat{\varphi})'_n - (\overline{\varphi})'_n \right]^2 d\Gamma +$$

$$+ \sum_I \left\{ \int_{\Gamma_I} \left(\hat{\varphi}^+ - \hat{\varphi}^- \right)^2 d\Gamma + W_I^2 \int_{\Gamma_I} \left[\left(\hat{\varphi}^+ \right)'_n + \left(\hat{\varphi}^- \right)'_n \right]^2 d\Gamma \right\} = \min$$

$\hat{\varphi}(x, y)$ - *Trefftz solution in each subregion*

$\overline{\varphi}$ - *given on* $\Gamma_\varphi \equiv \Gamma_1$

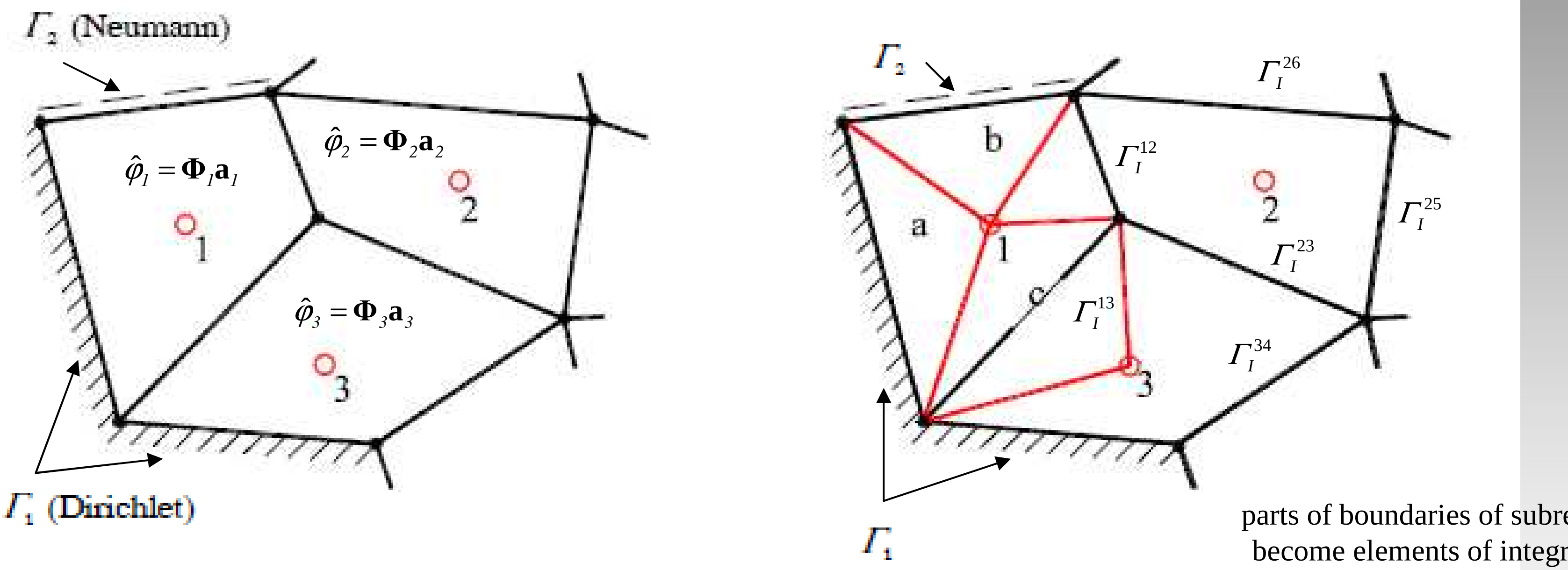
$(\overline{\varphi})_n{}'$ - *given on* $\Gamma_{qn} \equiv \Gamma_2$

$$\varphi(x, y) \approx \hat{\varphi}(x, y) = \sum_k a_k \Phi_k(x, y) \equiv \mathbf{\Phi a}$$

$$\mathbf{\Phi} \equiv \{\Phi_1, \Phi_2, \ldots .\Phi_K\}, \quad \mathbf{a}^T \equiv \{a_1, a_2, \ldots .a_K\}$$

Φ_k - *Trefftz trial functions – fulfilling homogeneous equation inside* Ω *including* Γ

Coupling of many subregions

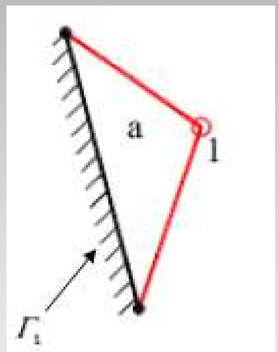


parts of boundaries of subregions become elements of integration

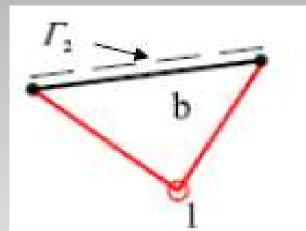
Coupling of many subregions

$$\delta J(\mathbf{a}) = \delta \mathbf{a}^{\mathrm{T}} \frac{\partial J}{\partial \mathbf{a}} = \delta \mathbf{a}^{\mathrm{T}} \left(\mathbf{K}\mathbf{a} - \mathbf{R} \right) = 0$$

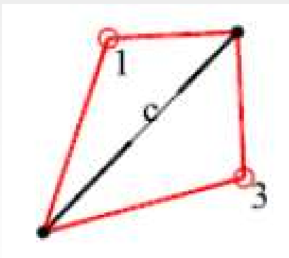
$$\delta J(\mathbf{a}) = \sum_{e=1}^{N_e} \delta J_e(\mathbf{a}_e) \qquad \delta J_e(\mathbf{a}_e) = \delta \mathbf{a}_e^{\mathrm{T}} \frac{\partial J_e(\mathbf{a}_e)}{\partial \mathbf{a}_e} = \delta \mathbf{a}_e^{\mathrm{T}} \left(\mathbf{r}_e^p + \mathbf{k}_e \mathbf{a}_e \right) = \delta \mathbf{a}_e^{\mathrm{T}} \mathbf{r}_e \longrightarrow \mathbf{r}_e = \mathbf{r}_e^p + \mathbf{k}_e \mathbf{a}_e$$



$$\mathbf{k}_{\mathrm{a}} = \sum_{\mu}^{\Gamma_1} \mathbf{\Phi}_1^T \mathbf{\Phi}_1$$



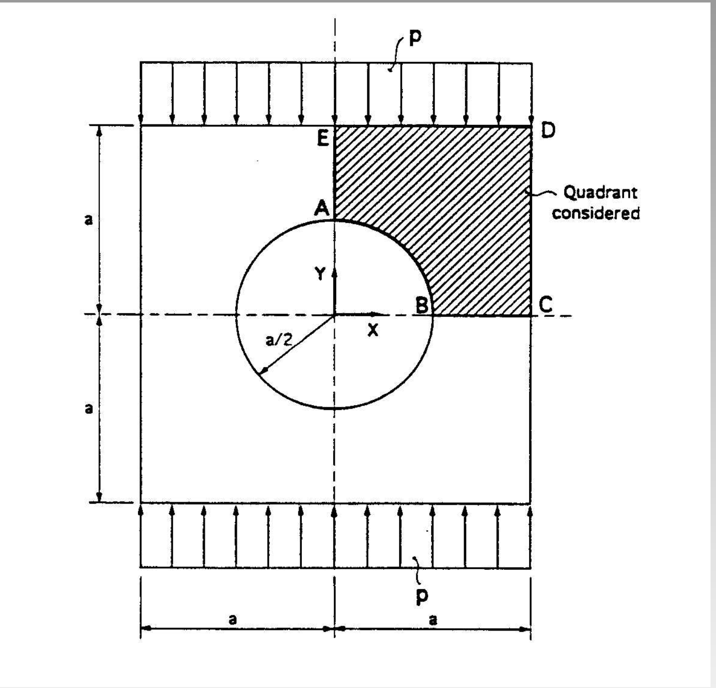
$$\mathbf{k}_{\mathrm{b}} = W^2 \sum_{\mu}^{\Gamma_2} \dot{\mathbf{\Phi}}_1^T \dot{\mathbf{\Phi}}_1$$



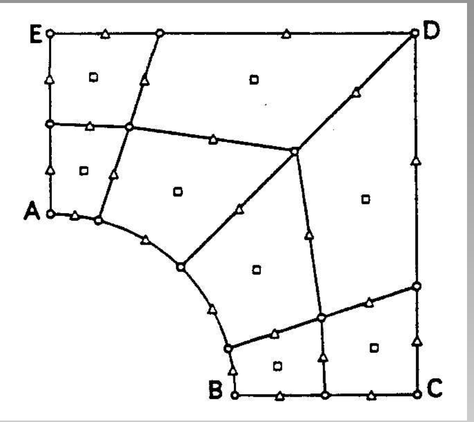
$$\mathbf{k}_{\mathrm{c}} = \begin{bmatrix} \sum_{\mu}^{\Gamma_{13}} \left[\mathbf{\Phi}_1^T \mathbf{\Phi}_1 + W^2 \dot{\mathbf{\Phi}}_1^T \dot{\mathbf{\Phi}}_1 \right] & \sum_{\mu}^{\Gamma_{13}} \left[-\mathbf{\Phi}_1^T \mathbf{\Phi}_3 + W^2 \dot{\mathbf{\Phi}}_1^T \dot{\mathbf{\Phi}}_3 \right] \\ \sum_{\mu}^{\Gamma_{13}} \left[-\mathbf{\Phi}_3^T \mathbf{\Phi}_1 + W^2 \dot{\mathbf{\Phi}}_3^T \dot{\mathbf{\Phi}}_1 \right] & \sum_{\mu}^{\Gamma_{13}} \left[\mathbf{\Phi}_3^T \mathbf{\Phi}_3 + W^2 \dot{\mathbf{\Phi}}_3^T \dot{\mathbf{\Phi}}_3 \right] \end{bmatrix}$$

Numerical example (J. Jirousek, A. Wróblewski, 1994):

Compression of a perforated panel



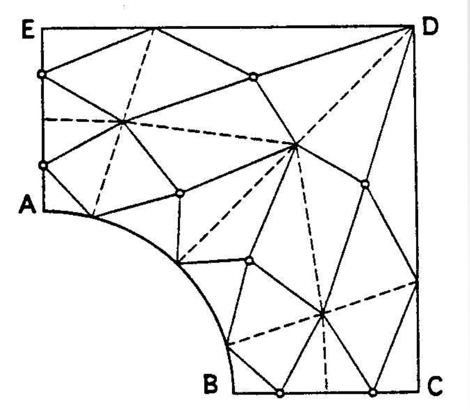
Conventional p-element mesh



232 DOF

(p=5)

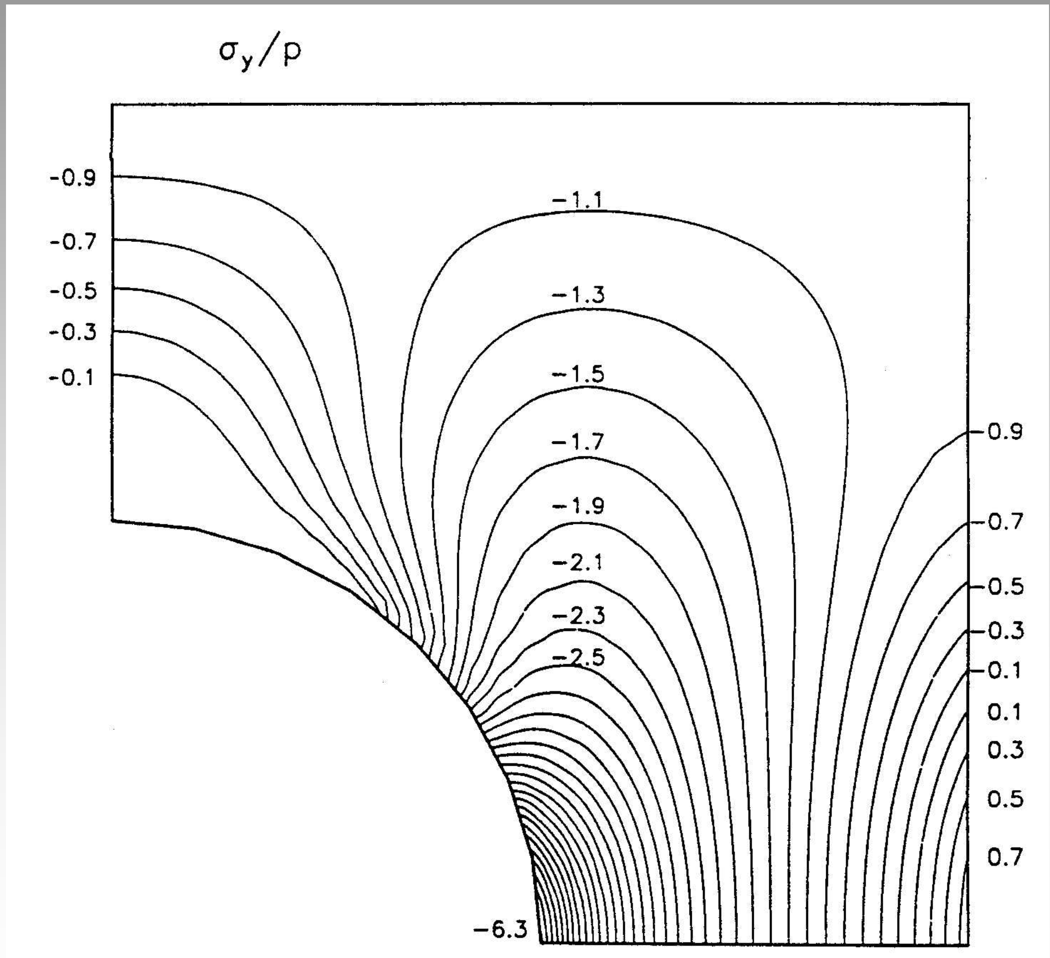
T-element mesh



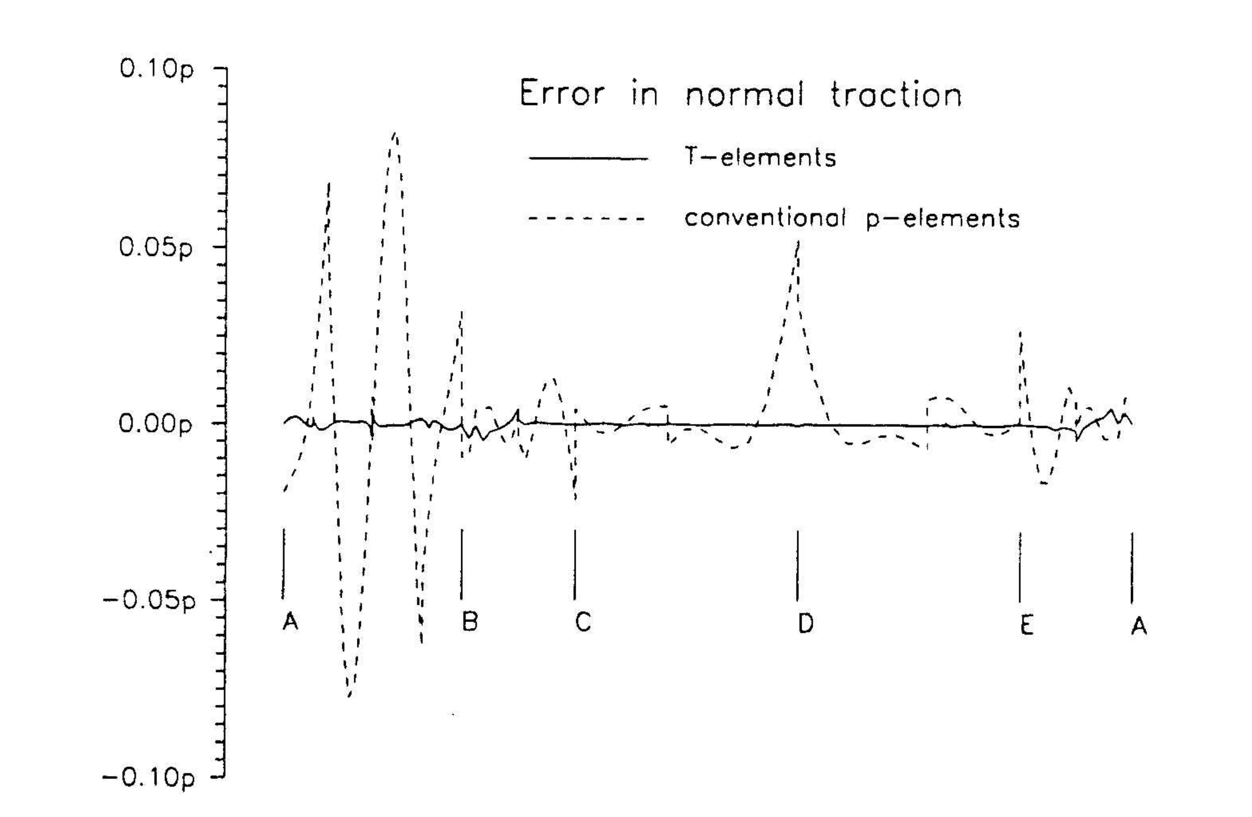
228 DOF

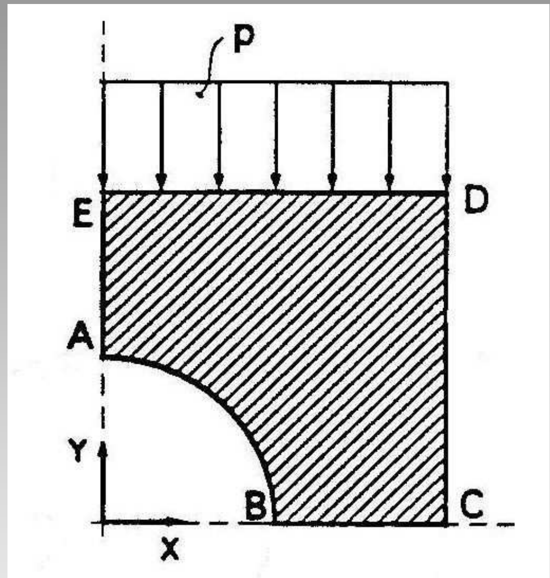
(p=9)

Numerical example: compression of a perforated panel

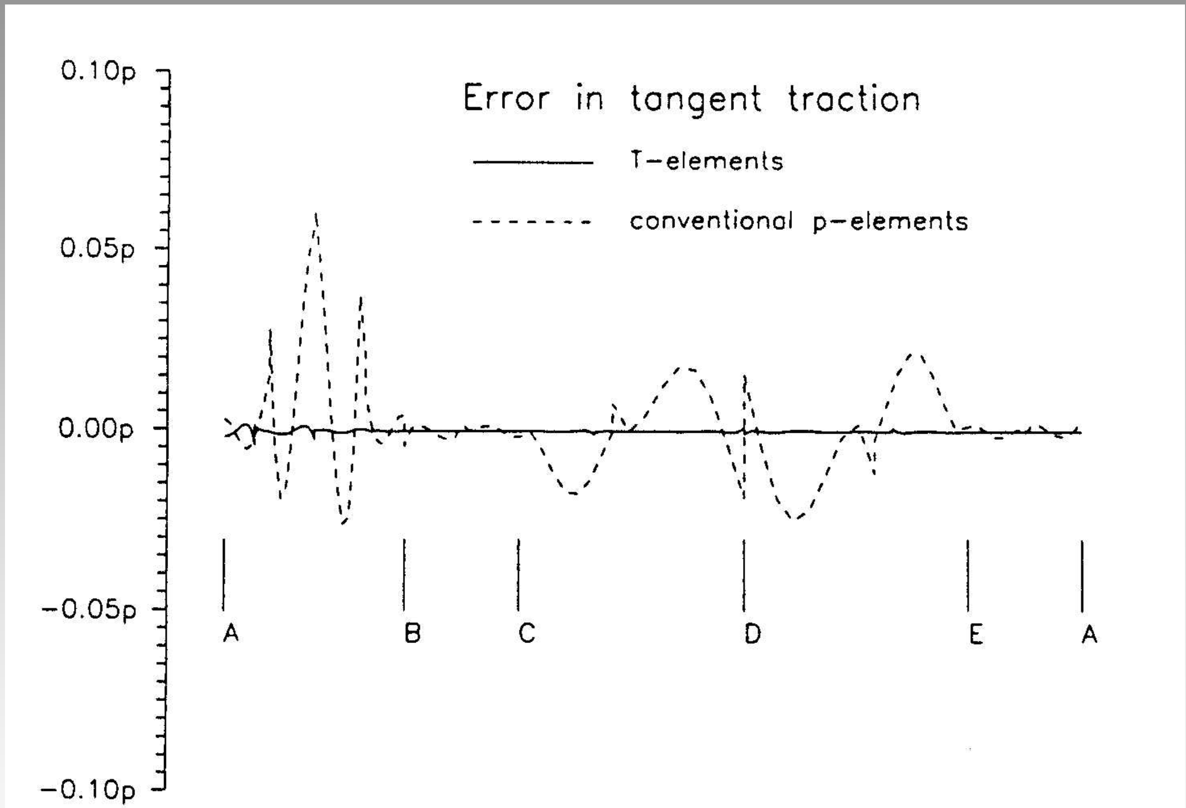


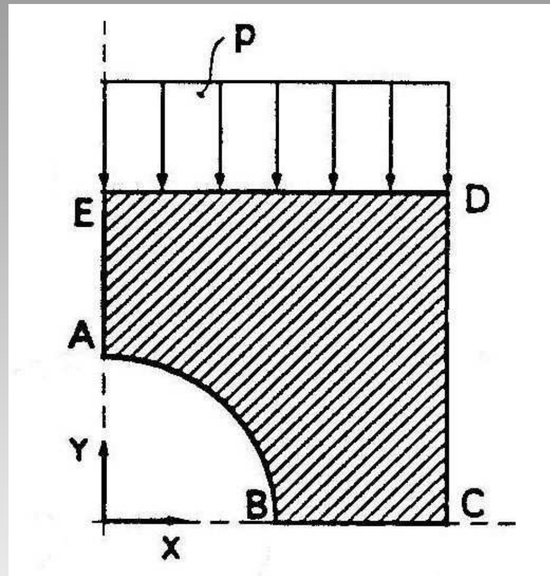
Numerical example: compression of a perforated panel





Numerical example: compression of a perforated panel



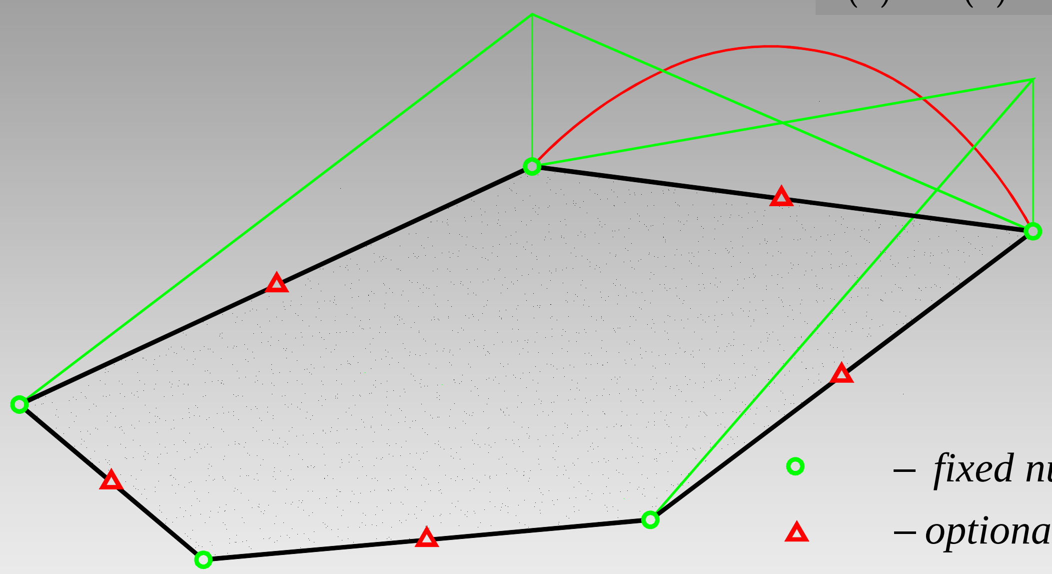


Hybrid-Trefftz displacement *p*-element (HT-D)

J. Jirousek 1977-1998

$$\tilde{N}_1^m = 1-\xi^2$$
$$\tilde{N}_2^m = \xi(1-\xi^2)$$
$$\tilde{N}_3^m = \xi^2(1-\xi^2)$$
$$\ldots\ldots\ldots\ldots\ldots$$

$$\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{N}}(\mathbf{x})\,\mathrm{d} \qquad \mathbf{x} \in \boldsymbol{\Gamma}^e$$



○ – *fixed number of DOF*

△ – *optional number* (M) *of DOF*

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^{\mathrm{p}}(\mathbf{x}) + \mathbf{N}(\mathbf{x})\mathbf{c} \qquad \mathbf{x} \in \boldsymbol{\Omega}^{\mathbf{e}}$$

N – matrix of Trefftz functions

$$\mathbf{t}(\mathbf{x}) = \mathbf{t}^{\mathrm{p}}(\mathbf{x}) + \mathbf{T}(\mathbf{x})\mathbf{c} \qquad \mathbf{x} \in \boldsymbol{\Gamma}^{\mathbf{e}}$$

T-complete system for 2D elasticity

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} u^p \\ v^p \end{Bmatrix} + \sum_{j=1}^{l} \begin{Bmatrix} N_{uj}^m \\ N_{vj}^m \end{Bmatrix} \cdot c_j^m$$

$$\mathbf{N}_{j+1}^m = \begin{Bmatrix} N_{u\ j+1}^m \\ N_{v\ j+1}^m \end{Bmatrix} = \begin{Bmatrix} \operatorname{Re} Z_{1k} \\ \operatorname{Im} Z_{1k} \end{Bmatrix} \qquad \mathbf{N}_{j+4}^m = \begin{Bmatrix} \operatorname{Re} Z_{4k} \\ \operatorname{Im} Z_{4k} \end{Bmatrix}$$

$$\mathbf{N}_{j+2}^m = \begin{Bmatrix} \operatorname{Re} Z_{2k} \\ \operatorname{Im} Z_{2k} \end{Bmatrix}$$

$$\mathbf{N}_{j+3}^m = \begin{Bmatrix} \operatorname{Re} Z_{3k} \\ \operatorname{Im} Z_{3k} \end{Bmatrix}$$

$$Z_{1k} = (3-\nu)iz^k + (1+\nu)kiz\bar{z}^{k-1},$$

$$Z_{2k} = (3-\nu)z^k - (1+\nu)kz\bar{z}^{k-1},$$

$$Z_{3k} = (1+\nu)i\bar{z}^k,$$

$$Z_{4k} = -(1+\nu)\bar{z}^k.$$

where: $z = x + iy, \quad \bar{z} = x - iy \qquad k = 0,1,2,...$

(notice: for k=0 there are only 2 independent functions)

T-complete system for plate bending problem

$$D \cdot \nabla^4 w = p \qquad w = w^p + \sum_{j=1}^{l} N_{wj}^b \cdot c_j^b$$

$$N_{j+1}^b = r^2 \cdot \operatorname{Re} z^k \qquad N_{j+2}^b = r^2 \cdot \operatorname{Im} z^k$$

$$N_{j+3}^b = \operatorname{Re} z^{k+2} \qquad N_{j+4}^b = \operatorname{Im} z^{k+2} \qquad k = 0,1,2,\ldots$$

where:

$$r^2 = x^2 + y^2 \qquad z = x + iy$$

Element of HT-D type

Fitting of internal solution to frame

$$\int_{\boldsymbol{\Gamma}^{\mathbf{e}}} \delta \mathbf{t}^{\mathrm{T}} (\mathbf{u} - \tilde{\mathbf{u}}) \, d\Gamma = 0$$

$$\delta \mathbf{t}^{\mathrm{T}} = \delta \mathbf{c}^{\mathrm{T}} \mathbf{T}^{\mathrm{T}} \quad \Rightarrow \quad \int_{\Gamma e} \mathbf{T}^{\mathrm{T}} \left(\mathbf{u} - \tilde{\mathbf{u}} \right) \mathrm{d}\Gamma = 0 \quad \Rightarrow \quad \mathbf{c} = \mathbf{c}\left(\mathbf{d}\right)$$

Equivalency of wirtual work

$$\int_{\boldsymbol{\Gamma}^{\mathbf{e}}} \delta \tilde{\mathbf{u}}^{\mathbf{T}} \mathbf{t} \; d\Gamma = \int_{\boldsymbol{\Gamma}_{\mathbf{t}}^{\mathbf{e}}} \delta \tilde{\mathbf{u}}^{\mathbf{T}} \bar{\mathbf{t}} \; d\Gamma + \delta \mathbf{d}^{\mathbf{T}} \mathbf{r}$$

$$\mathbf{r} - \mathbf{r}^{\mathbf{p}} = \mathbf{k} \, \mathbf{d}$$

$$\mathbf{k} = \mathbf{G}^{\mathrm{T}} \mathbf{H}^{-1} \mathbf{G}$$

k – symmetric stiffnes matrix
d – vector of degrees of freedom

Element of HT-LS type

Fitting of internal solution to frame

$$\int_{\Gamma^e} (\mathbf{u} - \tilde{\mathbf{u}})^2 \, d\Gamma \xrightarrow[\mathbf{u}]{} \min$$

$$\mathbf{c} = \mathbf{c}(\mathbf{d})$$

Equivalency of wirtual work

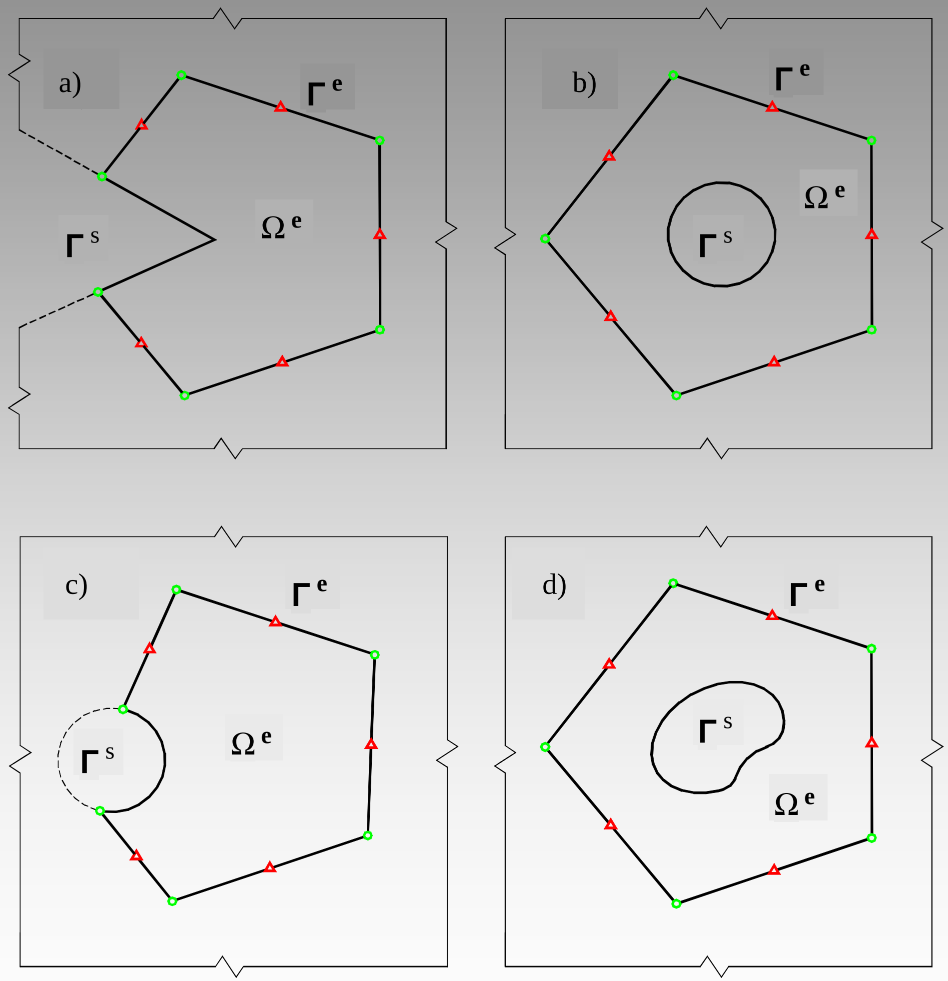
$$\int_{\Gamma^e} \delta \mathbf{u}^{\mathrm{T}} \mathbf{t} \, d\Gamma = \int_{\Gamma^t} \delta \mathbf{u}^{\mathrm{T}} \bar{\mathbf{t}} \, d\Gamma + \delta \mathbf{d}^{\mathrm{T}} \mathbf{r}$$

$$\mathbf{r} - \mathbf{r}^{\mathbf{p}} = \mathbf{k} \, \mathbf{d}$$

$$\mathbf{k} = \mathbf{F}^{\mathrm{T}} \mathbf{H} \mathbf{F}$$

k – symmetric stiffnes matrix
d – vector of degrees of freedom

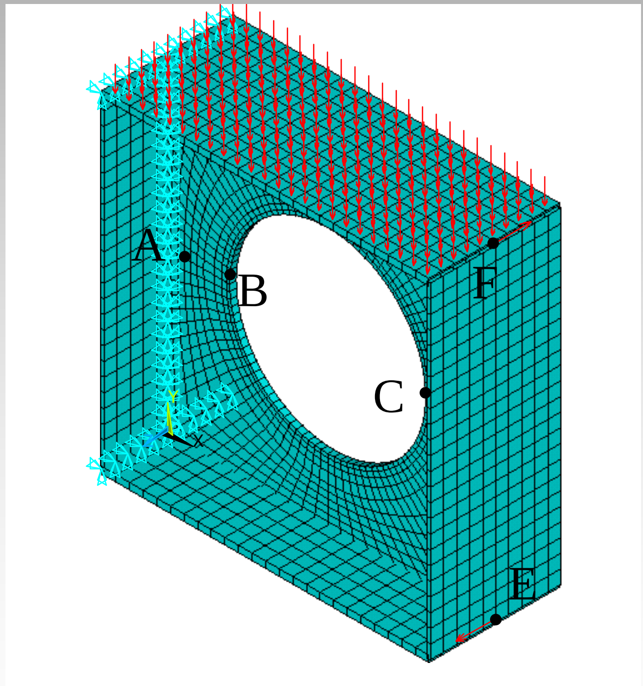
Different types of special purpose T-elements

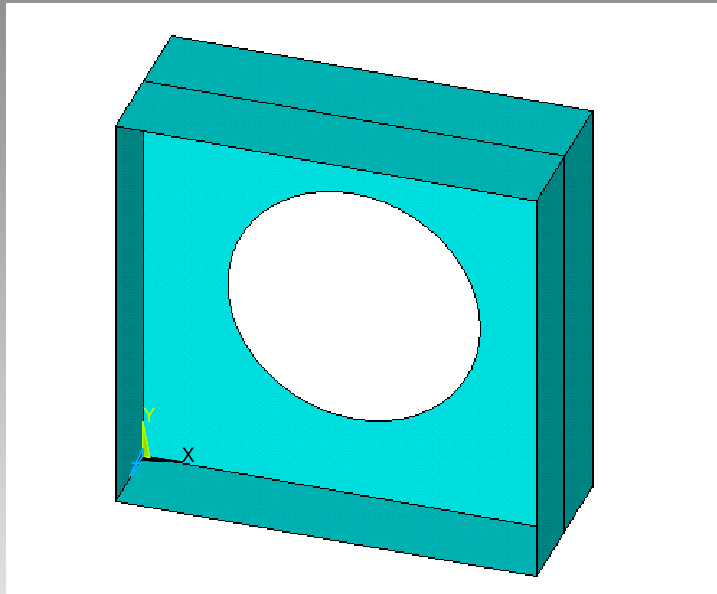


Folded plate structure — test for T-element features

T-elements: N_{ACT}=456

ANSYS: N_{ACT}=11981

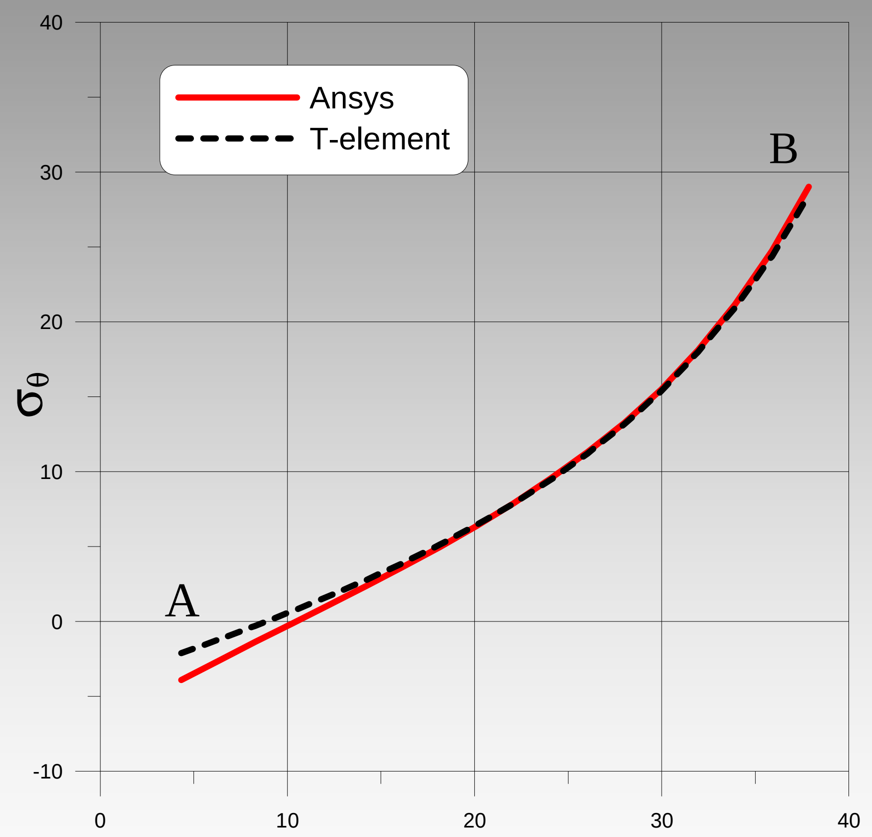


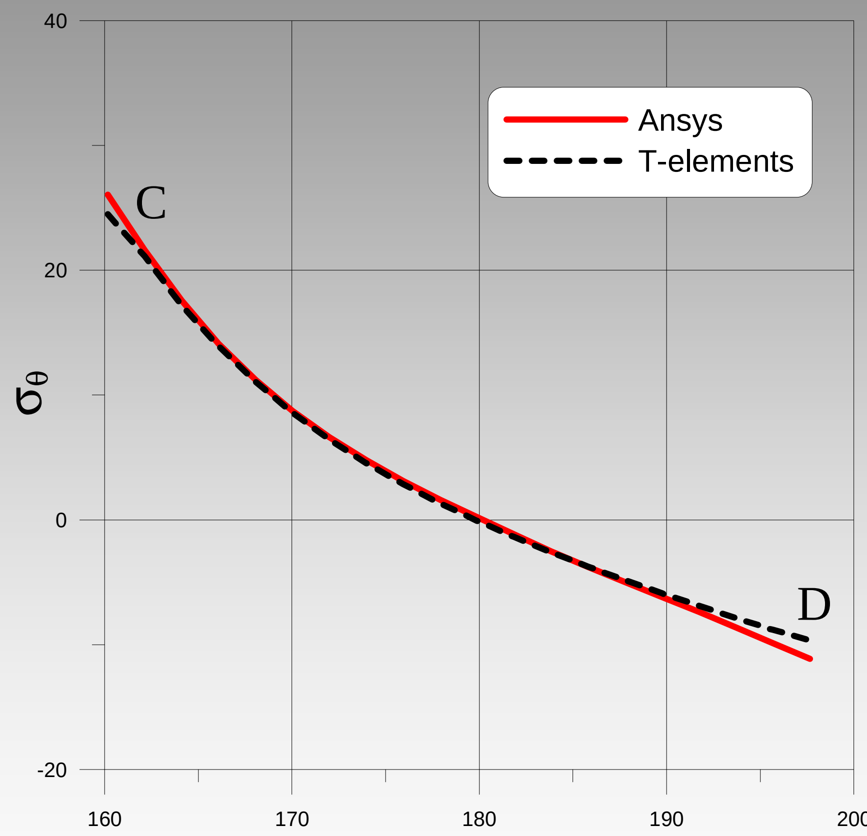


Loadings:
external pressure and nodal forces:

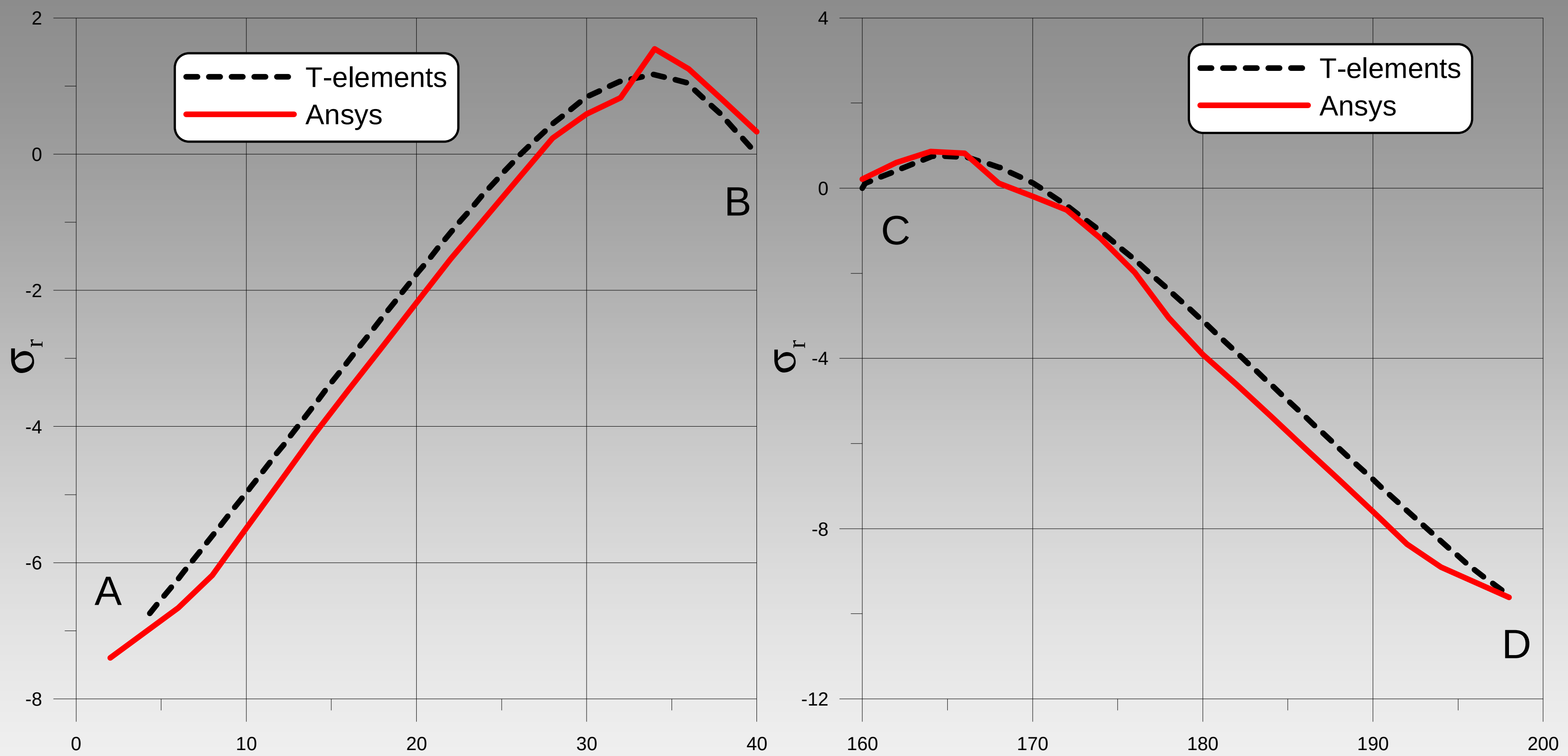
- on top panel, p_y = -0.5 [N/mm^2]
- on bottom panel, p_y= 0.5 [N/mm^2]
- in point E, F_x= -100 [N], F_z= 2500 [N]
- in point F, F_x= 100 [N], F_z= -2500 [N]

Results of tests for HT-D element - distribution of circumferential stressses σ_θ along lines AB and CD





Results of tests for HT-D element - radial stresses σ_r along lines AB, CD



Max. and min. value of radial stress σ_r in [MPa] along the boundary of the hole

	ANSYS	T-elements
σ_{rr}^{min}	-0.5428	$\times 10^{-13}$
σ_{rr}^{max}	0.4209	$\times 10^{-13}$

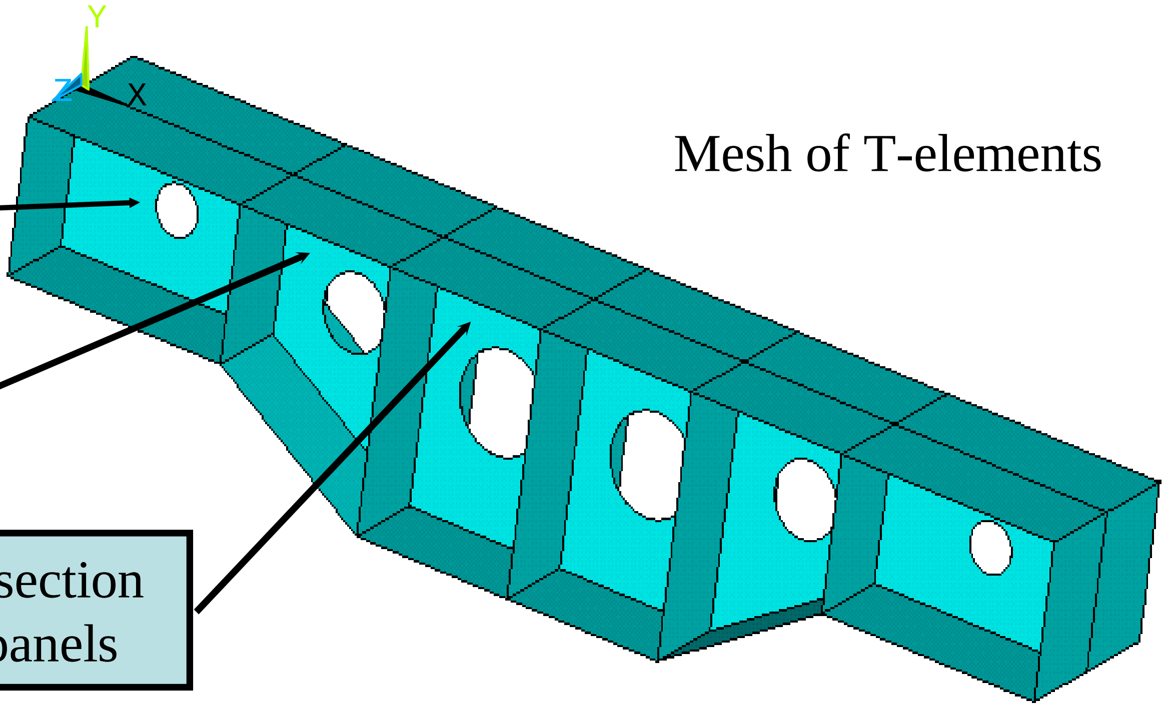
Simply supported at both ends plate girder with 6 circular openings

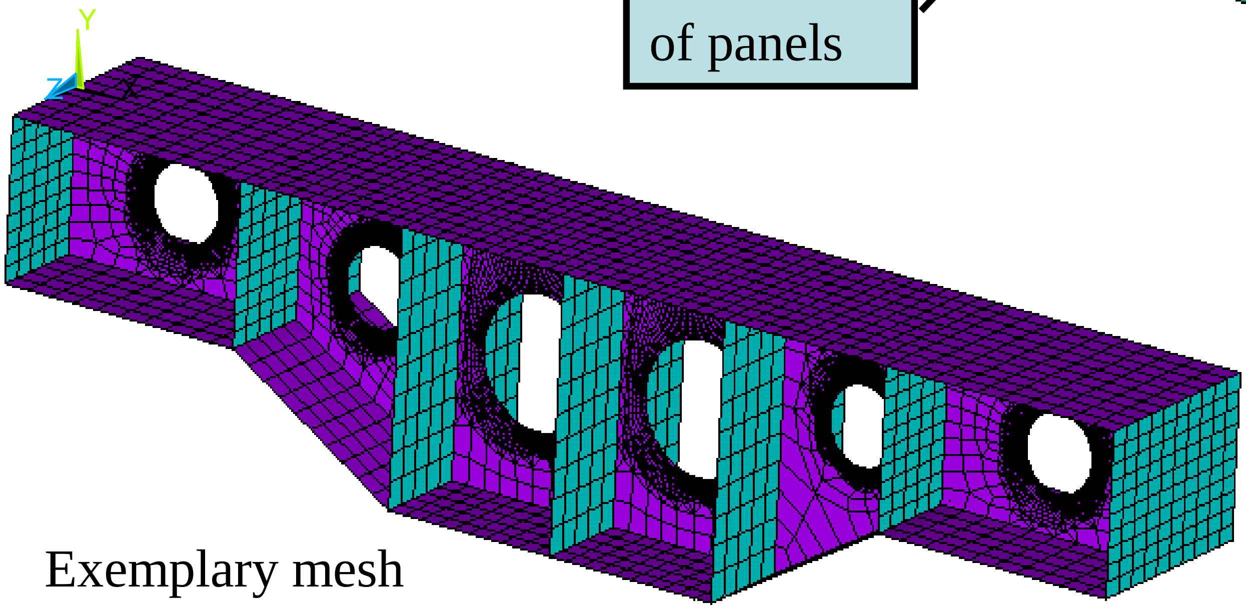
1st section of panels

2nd section of panels

3rd section of panels

Mesh of T-elements





Exemplary mesh in ANSYS

Local mesh refinement introduced in the vicinity of holes

Thank you for your attention !