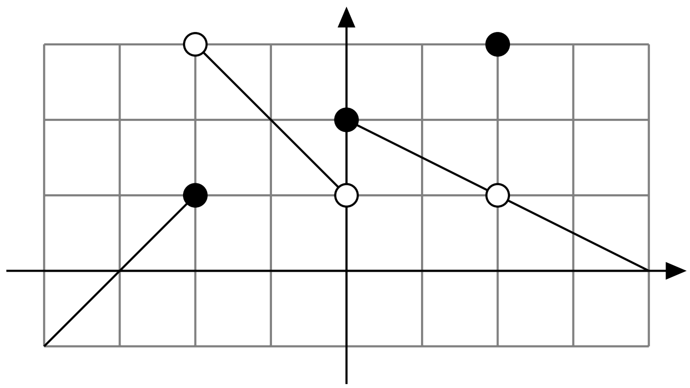
Read the limit from a graph. Let $f(x)$ be a function. The limit of $f(x)$ at $x = a$ means the value of $f(x)$ when x approaches to a (and never equal to a). There are two ways of approaching–from the left or from the right. These are called the left and the right limits; *when they are the same, we say the limit of $f(x)$ exists at $x = a$ and equals this common value.* For example, if the graph below is $y = f(x)$, then $\lim_{x\to -2^-} f(x) = 1$ and $\lim_{x\to -2^+} f(x) = 3$, so $\lim_{x\to -2} f(x)$ does not exist; in contrast, $\lim_{x\to 2^-} f(x) = 1 = \lim_{x\to 2^+} f(x)$, so $\lim_{x\to 2} f(x) = 1$. (The fact that $f(2) = 3$ does not change the limit.)



For the same graph, try to find out $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ and $\lim_{x\to -1^-} f(x)$, $\lim_{x\to -1^+} f(x)$, $\lim_{x\to -1} f(x)$. (The answer should be 1, 2, DNE, and 2, 2, 2.)

Composition. When there is another function $g(x)$ which is continuous (most of the functions you have seen is continuous), then *the limit can move inside-outside freely.* For example, let $f(x)$ be the same function given in the graph, and $g(x) = x^2 + x + 2$. Then what is $\lim_{x\to -1} g(f(x))$ and $\lim_{x\to 2} g(f(x))$? You can do $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$ to find the answer. (They should be 8 and 4.)

Infinity. A limit can also be ∞ or $-\infty$. For example, $\lim_{x\to 1^+} \frac{x+1}{x-1}$ is approaching to $\frac{2}{0^+}$, so the answer would be ∞; the 0^+ comes from $1^+ - 1$ is a small positive number. A bit complicated case will be $\lim_{x\to 1^+} \frac{-x}{(x-5)(1-x)}$, which approaches to $\frac{-1}{(-4)(0^-)}$, so it would be $-\infty$; the 0^- comes from $1 - 1^+$ is a small negative number. Combining all we have now, can you do $\lim_{x\to -2^-} \frac{f(x)}{f(x)-1}$ or $\lim_{x\to 0^+} \frac{f(x)-5}{f(x)-2}$, provided that $f(x)$ is the same function in the graph? (They are $-\infty$ and ∞, but why?)

Recall that x can also approach to ∞ or $-\infty$. Think about $\lim_{x\to\infty} \frac{2x^3+x}{5x^3+x^2} = \lim_{x\to\infty} \frac{2+(1/x^2)}{5+(1/x)}$, which equals $\frac{2}{5}$. How about $\lim_{x\to\infty} \frac{3x^{100}+x}{4x^{100}+x^2}$ or $\lim_{x\to -\infty} \frac{17x^{2016}+x^{600}}{19x^{2016}+x^{500}}$?

$$\text{Find out } \lim_{x\to -\infty} f\left(\frac{-x^{30}+4x+3}{x^{30}-5x^4+6}\right), \text{ where } f \text{ is the function in the graph.}$$

Find the limit by algebra. There are only a few question types: cancel the zeros, make up the conjugate, or the $\frac{\sin(x)}{x}$ type. Try the following:

(i) $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+x-6}$ **(iii)** $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x^2-2x}$ **(v)** $\lim_{x\to 0}\frac{\sin(10x)}{7x}$

(ii) $\lim_{x\to 3}\frac{x^2-4x+3}{x^2-x-6}$ **(iv)** $\lim_{x\to 1}\frac{x^2-2x+1}{3-\sqrt{x+8}}$ **(vi)** $\lim_{x\to 0}\frac{\sin(x^2+2x)}{x}$

Answers: **(i)** 0 **(ii)** $\frac{2}{5}$ **(iii)** $\frac{1}{8}$ **(iv)** 0 **(v)** $\frac{10}{7}$ **(vi)** 2

Asymptotes and discontinuity removal. If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$ for some finite number b, *then* $y = b$ *is a horizontal asymptote.* If $\lim_{x\to a} f(x)$ is of the form of $\frac{\text{nonzero value}}{0^+ \text{ or } 0^-}$, *then* $x = a$ *is a vertical asymptote.* If $\lim_{x\to a} f(x) = b$, then *by setting* $f(a) = b$ *the function becomes continuous at* $x = a$. Try the following:

(i) For $f(x) = \frac{2x^2+2x-4}{3x^2-12x+9} = \frac{2(x-1)(x+2)}{3(x-1)(x-3)}$, find all horizontal/vertical asymptote(s). Is $x = 1$ a vertical asymptote? Can you remove the discontinuity at $x = 1$ by setting $f(1)$ as some value?

(ii) For $f(x) = \frac{3x^2+3x-6}{4x^2-8x-12} = \frac{3(x-1)(x+2)}{4(x+1)(x-3)}$, find all horizontal/vertical asymptote(s).

[Note that you need to do the factorization by yourself in the exam.]

Answers: **(i)** Horizontal: $y = \frac{2}{3}$, vertical: $x = 3$. The line $x = 1$ is not an asymptote, since the value of $f(x)$ does not goes to infinity. By setting $f(1) = -1$, the function becomes continuous at 1. **(ii)** Horizontal: $y = \frac{3}{4}$, vertical: $x = -1$ and $x = 3$.

Average rate of change. When $f(x)$ describe a quantity (height, density, money, etc.) that depends on another quantity x (location, time, etc.), *the average rate of change between* $x = a$ *and* $x = b$ *is*

$$\frac{f(b)-f(a)}{b-a} \text{ or } \frac{f(a+h)-f(a)}{h}, \text{ where } h = b-a.$$

[Notice that there is nothing to do with the derivative so far.]

Definition of the derivative. Let $f(x)$ be a function. *The derivative of* $f(x)$ *is*

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Try finding the derivative of **(i)** x^2 **(ii)** $\frac{1}{x+1}$ **(iii)** $\sqrt{2x+3}$ by the limit definition. Do they agree with what you get by the differential rules?

Finding the limit through the derivative. Limits defines derivatives, but derivatives can also help on finding limits. Try:

(i) The limit $\lim_{h\to 0}\frac{3(2+h)^2-3\cdot 2^2}{h}$ equals $f'(a)$ for some function f and some value a. What are they, and what is the limit?

(ii) The limit $\lim_{x\to 3}\frac{\sqrt{2x+3}-\sqrt{2\cdot 3+3}}{x-3}$ equals $f'(a)$ for some function f and some value a. What are they, and what is the limit?

Answers: **(i)** $f(x) = 3x^2$, $a = 2$, limit is 12. **(ii)** $f(x) = \sqrt{2x+3}$, $a = 3$, limit is $\frac{1}{3}$.

Differentiation for fun. Find the derivatives of the functions below.

1. $-x^2 + 3$
2. $x^2 + x + 8$
3. $5t^3 - 3t^5$
4. $3z^{-2} - \frac{1}{z}$
5. $4 - 2x - x^{-3}$
6. $\frac{1}{3s^2} - \frac{5}{2s}$
7. $(3 - x^2)(x^3 - x + 1)$
8. $(2x + 3)(5x^2 - 4x)$
9. $(x^2 + 1)(x + 5 + \frac{1}{x})$
10. $(1 + x^2)(x^{3/4} - x^{-3})$
11. $\frac{2x+5}{3x-2}$
12. $\frac{4-3x}{3x^2+x}$
13. $\frac{x^2-x}{x+0.5}$
14. $\frac{t^2-1}{t^2+t-2}$
15. $(1 - t)(1 + t^2)^{-1}$
16. $\frac{\sqrt{s}-1}{\sqrt{s}+1}$
17. $\frac{5x+1}{2\sqrt{x}}$
18. $\frac{1}{(x^2-1)(x^2+x+1)}$
19. $\frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}}$
20. $\frac{e^s}{s}$

Answers without simplification.

1. $-2x$
2. $2x+1$
3. $15t^2 - 15t^4$
4. $-6z^{-3} + z^{-2}$
5. $-2 + 3x^{-4}$
6. $-\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2}$
7. $(-2x)(x^3 - x + 1) + (3 - x^2)(3x^2 - 1)$
8. $(2)(5x^2 - 4x) + (2x+3)(10x^2 - 4)$
9. $(2x)(x + 5 + \frac{1}{x}) + (x^2+1)(1 - x^{-2})$
10. $(2x)(x^{3/4} - x^{-3}) + (1 + x^2)(\frac{3}{4}x^{-1/4} + 3x^{-4})$
11. $\frac{(2)(3x-2)-(2x+5)(3)}{(3x-2)^2}$
12. $\frac{(-3)(3x^2+x)-(4-3x)(6x+1)}{(3x^2+x)^2}$
13. $\frac{(2x-1)(x+0.5)-(x^2-x)(1)}{(x+0.5)^2}$
14. $\frac{(2t)(t^2+t-2)-(t^2-1)(2t+1)}{(t^2+t-2)^2}$
15. $\frac{(-1)(1+t^2)-(1-t)(2t)}{(1+t^2)^2}$
16. $\frac{(\frac{1}{2}s^{-1/2})(\sqrt{s}+1)-(\sqrt{s}-1)(\frac{1}{2}s^{-1/2})}{(\sqrt{s}+1)^2}$
17. $\frac{(5)(2\sqrt{x})-(5x+1)(x^{-1/2})}{(2\sqrt{x})^2}$
18. $\frac{-(2x)(x^2+x+1)-(x^2-1)(2x+1)}{(x^2-1)^2(x^2+x+1)^2}$
19. $-1.4z^{-2.4} - \frac{\pi}{2}z^{-3/2}$
20. $\frac{(e^s)(s)-(e^s)(1)}{s^2}$

Find the derivative:

1. $2(8x-1)^3$
2. $\cos(3x^2+1)$
3. e^{-5x}
4. $(4x+3)^4(x+1)^{-3}$
5. $\frac{4}{3\pi}\sin(3t)+\frac{4}{5\pi}\cos(5t)$
6. $\cos(\sin(x^2))$
7. $-\sec(x^2+7x)$
8. $\left(1-\frac{x}{7}\right)^{-7}$
9. $5(\cos(x))^{-4}$
10. $\sin^3(x^4)$
11. e^{x^4} (notice that this is different from $(e^x)^4$, which is e^{4x})
12. $e^{4\sqrt{x}+x^2}$
13. $xe^{-x}+e^{3x}$
14. $\tan^2(\sin^3(t))$
15. $\sin(x^2e^x)$

Find the tangent line. For a given function $f(x)$, the tangent line of f at $x=a$ is a line with $m=f'(a)$ and passing through the point $(a,b)=(a,f(a))$. For the function equal to the formula in 1,2,3 above, find its tangent line at $x=1$.

Find the value of $F'(x)$ with limited information. Let $F(x)=f(g(x))$ for two functions f and g. Suppose all we know is $f(2)=5$, $f'(2)=4$, $g(1)=2$, $g'(1)=3$. Can you find $F'(1)$?

Tangent line for a composite function. Can you find the tangent line of $F(x)$ described above at $x=1$?

Answers without simplification.

1. $6(8x-1)^2 \cdot 8$
2. $-\sin(3x^2+1) \cdot (6x)$
3. $e^{-5x} \cdot (-5)$
4. $[4(4x+3)^3 \cdot 4](x+1)^{-3} + (4x+3)^4[-3(x+1)^{-4}]$
5. $\frac{4}{3\pi}\cos(3t) \cdot 3 - \frac{4}{5\pi}\sin(5t) \cdot 5$
6. $-\sin(\sin(x^2)) \cdot \cos(x^2) \cdot (2x)$
7. $-\sec(x^2+7x)\tan(x^2+7x) \cdot (2x+7)$
8. $-7\left(1-\frac{x}{7}\right)^{-8} \cdot \left(-\frac{1}{7}\right)$
9. $-20(\cos(x))^{-5} \cdot (-\sin(x))$
10. $3\sin^2(x^4) \cdot \cos(x^4) \cdot (4x^3)$
11. $e^{x^4} \cdot (4x^3)$
12. $e^{4\sqrt{x}+x^2} \cdot (2x^{-\frac{1}{2}} + 2x)$
13. $e^{-x} - xe^{-x} + 3e^{3x}$
14. $2\tan(\sin^3(t)) \cdot \sec^2(\sin^3(t)) \cdot 3\sin^2(t) \cdot \cos(t)$
15. $\cos(x^2e^x) \cdot (2xe^x + x^2e^x)$

Find the tangent line. Recall the line with slope m and passing the point (a, b) is $y = m(x-a)+b$.

1. $y = 2352(x-1) + 686$
2. $y = -6\sin(4)(x-1) + \cos(4)$
3. $y = -5e^{-5}(x-1) + e^{-5}$

Find the value of $F'(x)$ with limited information.

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$F'(1) = f'(g(1))g'(1) = f'(2) \cdot 3 = 4 \cdot 3 = 12$$

Tangent line for a composite function. The tangent line passes through $(1, F(1)) = (1, f(g(1))) = (1, f(2)) = (1, 5)$. The slope is $F'(1) = 12$. So the line is $y = 12(x-1)+5$.

The limit definition. The derivative of a function $f(x)$ at $x = a$ is defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This definition helps you to find the derivative, and also allows you to find the limit through the differential rules. (See the Limit review sheet.)

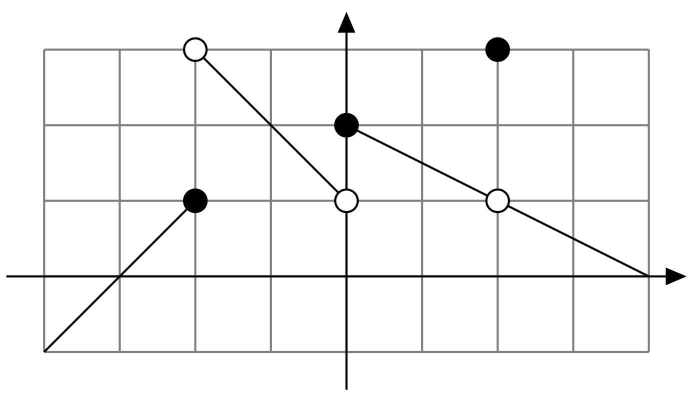
Meaning of the derivative. On the graph $y = f(x)$, the derivative is *the slope of the tangent line*; when $f(x)$ describes the motion and x is the time, then $f'(x)$ *is the velocity and* $f''(x)$ *is the acceleration*. Recall that *the speed is* $|f'(x)|$.

Read the derivative from a graph. The derivative can be considered as the *momentum*; *the derivative is positive or negative when the graph is increasing or decreasing, respectively.* The point where the momentum changes receive its derivative zero. In some cases one can read the exact value of the derivative from the graph. For example, for the function $f(x)$ shown below, find the following:

(i) $f(-3)$ **(iii)** $f(-1)$ **(v)** $f(2)$ **(vii)** $f(3)$

(ii) $f'(-3)$ **(iv)** $f'(-1)$ **(vi)** $f'(2)$ **(viii)** $f'(3)$

[Hint: Think about what the slope means.]
Answers: **(i)** 0 **(ii)** 1 **(iii)** 2 **(iv)** -1 **(v)** 3 **(vi)** DNE **(vii)** 0.5 **(viii)** $-\frac{1}{2}$



Work abstractly. Some differential rules allow you to find out the derivative by limited information. For example, we have

Product Rule If $f(x) = AB$, then $f'(x) = A'B + AB'$.

Quotient Rule If $f(x) = \frac{A}{B}$, then $f'(x) = \frac{A'B - AB'}{B^2}$.

Chain Rule If $F(x) = f(g(x))$, then $F'(x) = f'(g(x)) \cdot g'(x)$.

Let $f(x)$ be the function drawn in the graph. Try the following:

(i) For $F(x) = x^2 f(x)$, find $F'(-3)$.

(ii) For $F(x) = \frac{x^2}{f(x)}$, find $F'(-1)$.

(iii) For $F(x) = f(f(x) - 5)$, find $F'(-1)$.

Here are the solutions:

(i) $F'(x) = 2xf(x) + x^2 f'(x)$, so $F'(-3) = 2 \cdot (-3) \cdot f(-3) + (-3)^2 \cdot f'(-3) = 9$.

(ii) $F'(x) = \frac{2xf(x) - x^2 f'(x)}{(f(x))^2}$, so $F'(-1) = \frac{2 \cdot (-1) \cdot f(-1) - (-1)^2 \cdot f'(-1)}{(f(-1))^2} = -\frac{3}{4}$.

(iii) $F'(x) = f'(f(x) - 5) \cdot f'(x)$, so $F'(-1) = f'(f(-1) - 5) \cdot f'(-1) = (1) \cdot (-1) = -1$.

The question might also give you a lot of redundant information. But calm down, find those you need only. Here is an example.

Suppose $u(x), v(x)$ are two functions, and we know

$u(1) = \sqrt{2}$	$v(1) = 3$	$u(3) = \sqrt{5}$	$v(3) = \sqrt{13}$
$u'(1) = \sqrt{3}$	$v'(1) = \pi$	$u'(3) = \sqrt{7}$	$v'(3) = \sqrt{17}$
$u''(1) = 2$	$v''(1) = e$	$u''(3) = \sqrt{11}$	$v''(3) = \sqrt{19}$

(i) For $F(x) = u(x)v(x)$, find $F'(1)$.

(ii) For $F(x) = u(x)v(x)$, find $F''(1)$.

(iii) For $F(x) = u(x)/v(x)$, find $F'(3)$.

(iv) For $F(x) = u(v(x))$, find $F'(1)$.

(v) For $F(x) = u(v(x))$, find $F''(1)$.

Here are the solutions:

(i) $F'(x) = u'(x)v(x) + u(x)v'(x)$, so $F'(1) = (\sqrt{3})(3) + (\sqrt{2})(\pi)$.

(ii) $F''(x) = u''(x)v(x) + u'(x)v'(x) + u'(x)v'(x) + u(x)v''(x)$, so $F''(1) = (2)(3) + (\sqrt{3})(\pi) + (\sqrt{3})(\pi) + (\sqrt{2})(e)$.

(iii) $F'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$, so $F'(3) = \frac{(\sqrt{7})(\sqrt{13}) - (\sqrt{5})(\sqrt{17})}{13}$.

(iv) $F'(x) = u'(v(x)) \cdot v'(x)$, so $F'(1) = u'(v(1)) \cdot v'(1) = u'(3) \cdot v'(1) = (\sqrt{7})(\pi)$

(v) $F''(x) = u''(v(x)) \cdot v'(x) \cdot v'(x) + u'(v(x)) \cdot v''(x)$, so $F''(1) = u''(3) \cdot v'(1) \cdot v'(1) + u'(3) \cdot v''(1) = (\sqrt{11})(\pi)(\pi) + (\sqrt{7})(e)$.

Implicit differentiation with different symbols.

(i) For $u^3 v + u^2 v^2 + uv^3 = 3$, consider u as a function of v and find $\frac{du}{dv}$.

(ii) For $r\theta + \cos(r\theta) = 5$, consider r as a function of θ and find $\frac{dr}{d\theta}$.

Here are the solutions.

(i) Let $u' = \frac{du}{dv}$. Then $3u^2 u'v + u^3 + 2uu'v^2 + 2u^2 v + u'v^3 + 3uv^2 = 0$ and rearrange it.

(ii) Let $r' = \frac{dr}{d\theta}$. Then $r'\theta + r - \sin(r\theta) \cdot (r'\theta + r) = 0$ and rearrange it.

Inverse function. Let f be a function with $f(a) = b$. Then its inverse function is denoted as f^{-1} and $f^{-1}(b) = a$. Note that a function has the inverse only when the function passes the *horizontal test*. For example, $f(x) = x^2$ does not have the inverse, since you don't know $f^{-1}(4)$ should be 2 or -2. However, we can consider the function $f(x) = x^2$ only on $x \geq 0$, then it has the inverse $f^{-1}(x) = \sqrt{x}$. That is, always pick the positive square root.

Inverse exponential function. The function $f(x) = e^x$ means the x-th power of the constant e. The inverse will be $f^{-1}(x) = \ln(x)$. The value $\ln(x)$ is a number p such that e^p is x. Therefore, $\ln(1) = 0$ and $\ln(e) = 1$. Some properties of the natural logarithmic function are

(i) $\ln(e^x) = x$ **(iii)** $\ln(x) = \log_e(x)$ **(v)** $\ln(a \cdot b) = \ln(a) + \ln(b)$

(ii) $e^{\ln(x)} = x$ **(iv)** $\log_a(b) = \frac{\ln(a)}{\ln(b)}$ **(vi)** $\ln(a^b) = b\ln(a)$

Inverse trigonometric functions. All the trig functions do not pass the horizontal test. But by restricting the angle, we still can define their inverses as below.

- $\sin^{-1}(x)$ is the angle θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\sin(\theta) = x$.
- $\cos^{-1}(x)$ is the angle θ in $[0, \pi]$ with $\cos(\theta) = x$.
- $\tan^{-1}(x)$ is the angle θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $\tan(\theta) = x$.
- $\sec^{-1}(x)$ is the angle θ in $[0, \pi]$ but $\frac{\pi}{2}$ with $\sec(\theta) = x$.
- $\csc^{-1}(x)$ is the angle θ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ but 0 with $\csc(\theta) = x$.
- $\cot^{-1}(x)$ is the angle θ in $(0, \pi)$ with $\cot(\theta) = x$.

For example, both $\tan(\frac{\pi}{4})$ and $\tan(\frac{5\pi}{4})$ are 1, but $\tan^{-1}(1)$ is $\frac{\pi}{4}$ instead of $\frac{5\pi}{4}$. We mentioned that $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$. How about $\lim_{x\to\infty} \cot^{-1}(x)$ and $\lim_{x\to\infty} \cot^{-1}(x)$? [0 and π]

Derivative of inverse function. When $f(a) = b$ and $f^{-1}(b) = a$, then $f'(a) \cdot f^{-1\prime}(b) = 1$. Here $f^{-1\prime}$ is an awful notation; it is acceptable, but $(f^{-1})'$ will be clearer. By this formula, we can derive $[\ln(x)]' = \frac{1}{x}$. Also we have the following.

(i) $[\sin^{-1}(x)]' = \frac{1}{\sqrt{1-x^2}}$ **(iii)** $[\cos^{-1}(x)]' = -\frac{1}{\sqrt{1-x^2}}$ **(v)** $[\tan^{-1}(x)]' = \frac{1}{1+x^2}$

(ii) $[\csc^{-1}(x)]' = -\frac{1}{|x|\sqrt{x^2-1}}$ **(iv)** $[\sec^{-1}(x)]' = \frac{1}{|x|\sqrt{x^2-1}}$ **(vi)** $[\cot^{-1}(x)]' = -\frac{1}{1+x^2}$

Think about how to derive these formulas?

Applying the chain rules to them, you will see that the derivative of $\ln(u)$ is $\frac{1}{u} \cdot u'$, where u is a piece of function. Similarly, $[\tan^{-1}(u)]' = \frac{1}{1+u^2} \cdot u'$, and so on.

Try finding the derivative of followings:

(i) $\ln(x^2)$ **(iii)** $\ln(\sin(x))$ **(v)** $\tan^{-1}(x^2+1)$ **(vii)** $\tan^{-1}(\ln(x))$

(ii) $\ln(e^x)$ **(iv)** $\ln(\sin(x^2))$ **(vi)** $\tan^{-1}(x^3)$ **(viii)** $\tan^{-1}(\ln(x^2))$

The answers should be **(i)** $\frac{2}{x}$ **(ii)** 1 **(iii)** $\frac{\cos(x)}{\sin(x)}$ **(iv)** $\frac{\cos(x^2)\cdot 2x}{\sin(x^2)}$ **(v)** $\frac{2x}{1+(x^2+1)^2}$ **(vi)** $\frac{3x^2}{1+(x^3)^2}$ **(vii)** $\frac{1}{x(1+(\ln(x))^2)}$ **(viii)** $\frac{2}{x(1+(\ln(x^2))^2)}$

Logarithmic differentiation. The properties of $\ln(x)$ allow us the conduct what we called the *logarithmic differentiation*. For example, we may write 3^x as $e^{\ln(3^x)} = e^{x\ln(3)}$. Thus its derivative becomes $e^{x\ln(3)} \cdot \ln(3) = 3^x \ln(3)$. Also, $\log_a(x)$ can be written as $\frac{\ln(x)}{\ln(a)}$, so its derivative is $\frac{1}{x\ln(a)}$. Try the following

(i) 5^x **(iii)** x^x **(v)** $2^x \cdot 3^x$ **(vii)** $\log_x(x^x)$

(ii) 5^{x^2} **(iv)** $x^{\sin(x)}$ **(vi)** $2^{(3^x)}$ **(viii)** $\log_x(5)$

The answers should be: **(i)** $5^x \ln(5)$ **(ii)** $5^{x^2} \cdot \ln(5) \cdot (2x)$ **(iii)** $x^x[\ln(x) + 1]$ **(iv)** $x^{\sin(x)}[\cos(x)\ln(x) + \sin(x) \cdot \frac{1}{x}]$ **(v)** $6^x \ln(6)$ **(vi)** $2^{(3^x)} \ln(2) \cdot 3^x \ln(3)$ **(vii)** 1 **(viii)** $\frac{-\ln(5)\cdot\frac{1}{x}}{[\ln(x)]^2}$

Tangent line. Up to now, we are able to do the derivative of almost all functions. The technique of finding the tangent line is the same.

(i) Find the tangent line of $f(x) = \tan^{-1}(x^3)$ at $x = 1$.

(ii) Find the tangent line of $f(x) = e^x$ at $x = 0$.

(iii) Find the tangent line of $f(x) = \ln(x)$ at $x = 1$.

(iv) Find the tangent line of $f(x) = 2^x$ at $x = 0$.

The answers should be:

(i) Slope: $\frac{3}{2}$, point: $(1, \frac{\pi}{4})$, line: $y = \frac{3}{2}(x-1) + \frac{\pi}{4}$.

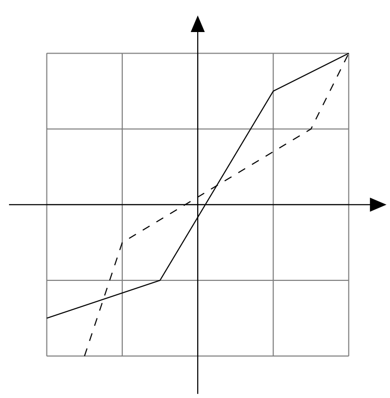
(ii) Slope: 1, point: $(0,1)$, line: $y = x + 1$.

(iii) Slope: 1, point: $(1,0)$, line: $y = x - 1$. [You can also obtain this by switching x and y in **(ii)**.]

(iv) Slope: $\ln(2)$, point: $(0,1)$, line: $y = \ln(2) \cdot x + 1$.

Tangent line of the inverse function. Suppose $f(a) = b$ and $f^{-1}(b) = a$. Then the tangent line for f at $x = a$ is $y = m(x-a) + b$ if and only if the tangent line for f^{-1} at $x = b$ is $y = \frac{1}{m}(x-b) + a$. Observe that x and y just switch their roles in the two formulas.

Graph of the inverse function. The graph of f and f^{-1} is symmetric to the 45° line. If f is the solid line below, what is the graph for f^{-1}?



Differentiation rules and related identities I

Why we care about the derivative is because it tells us the *slope* of the tangent line, or the *velocity*.

Basic Differential Rules:

1. $f(x) = c \to f'(x) = 0$, where c is a constant.
2. $f(x) = x^n \to f'(x) = nx^{n-1}$, where n can be any real number. [**Power Rule**]
3. $f(x) = e^x \to f'(x) = e^x$.
4. $f(x) = c_1 A(x) + c_2 B(x) \to f'(x) = c_1 A'(x) + c_2 B'(x)$, where c_1, c_2 are constants and A, B are functions. [**Linearity**]
5. $f(x) = AB \to f'(x) = A'B + AB'$, where A, B are two functions. [**Product Rule**]
6. $f(x) = \frac{A}{B} \to f'(x) = \frac{A'B - AB'}{B^2}$, where A, B are two functions. Notice it is a minus in the numerator. [**Quotient Rule**]
7. $F(x) = f(g(x)) \to F'(x) = f'(g(x)) \cdot g'(x)$. [**Chain Rule**]
8. $f(x) = \sin(x) \to f'(x) = \cos(x)$.
9. $f(x) = \cos(x) \to f'(x) = -\sin(x)$.

Then you can have:

1. $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} \to f'(x) = \sec^2(x)$.
2. $f(x) = \sec(x) = \frac{1}{\cos(x)} \to f'(x) = \sec(x)\tan(x)$.
3. $f(x) = \csc(x) = \frac{1}{\sin(x)} \to f'(x) = -\csc(x)\cot(x)$.
4. $f(x) = \cot(x) = \frac{\cos(x)}{\sin(x)} \to f'(x) = -\csc^2(x)$.

For example, when $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, then you may use the quotient rule with $A = \sin(x)$ and $B = \cos(x)$. Thus we have $A' = \cos(x)$ and $B' = -\sin(x)$ and then

$$f'(x) = \frac{A'B - AB'}{B^2} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

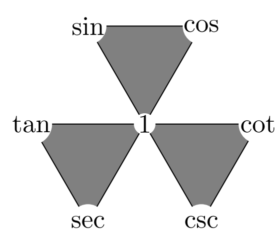
Notice that $\cos^2(x)$ means $\cos(x)\cos(x)$, yet $\cos^{-1}(x)$ is the *arc cosine* function (the inverse of cosine), which **is different** from $(\cos(x))^{-1} = \frac{1}{\cos(x)}$.

How power works:

1. $x^a \cdot x^b = x^{a+b}$. (No way to simplify $x^a + x^b$.)
2. $(x^a)^b = x^{ab}$.
3. $x^{-1} = \frac{1}{x}$.
4. $\sqrt{x} = x^{\frac{1}{2}}$.

For example, since $\sqrt{x} = x^{\frac{1}{2}}$, it's derivative is $\frac{1}{2}x^{-\frac{1}{2}}$. Similarly, since $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, it's derivative is $-\frac{1}{2}x^{-\frac{3}{2}}$.

The hexagon for trig functions:



Each diagonal line means a reciprocal relation, e.g. $\sin(x) = \frac{1}{\csc(x)}$. Each upside down triangle represents an identity, e.g. $\sin^2(x) + \cos^2(x) = 1$ and $\tan^2(x) + 1 = \sec^2(x)$.

More trig identities:

1. $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.
2. $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.
3. $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

Then you have:

1. $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$.
2. $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.
3. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
4. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1$.
5. $\sin(\frac{\theta}{2}) = \sqrt{\frac{1-\cos(\theta)}{2}}$
6. $\cos(\frac{\theta}{2}) = \sqrt{\frac{1+\cos(\theta)}{2}}$

Equation of a line. We all know that two points determine a line. But a point along with its slope is also enough to determine a line! Suppose a line has slope m and it passes a point (a, b). *Then the equation for this line is*

$$y = m(x - a) + b.$$

Find a point. Suppose $f(x)$ is a function. By the graph of $f(x)$, we mean the graph of $y = f(x)$. *This graph collects all the points of the form* $(x, f(x))$. For example, when $f(x) = x^2$, it contains many points such as $(0, 0)$, $(2, 4)$, or $(-3, 9)$; however, $(0, 1)$ is not a point of this graph, since $f(0)$ is 0 instead of 1.

For other equations used in the implicit differentiation, such as $x^2+y^2 = 9$. *A point* (a, b) *lies on its graph means the equation holds when you plug in* $x = a$ *and* $y = b$. For example, $(3, 0)$ and $(1, 2\sqrt{2})$ are on the graph of $x^2 + y^2 = 9$, yet $(1, 1)$ is not. This is because if you have $x = 1$ and $y = 1$, then

$$1^2 + 1^2 = 2 \neq 9.$$

Find the slope. This part is easy! When you have a function $f(x)$, *the slope of it at* $x = a$ *is* $f'(a)$. When you have a equation, then use the implicit differentiation to find out y' and plug in the desired x and y. *This value of* y' *is the slope.*

Find the point by x.

(i) For the function $f(x) = x^2 + \frac{1}{x}$, find the points on this graph with $x = 1, 2, 3$, respectively.

(ii) For the function $f(x) = \sin(x) + e^{x^2-x}$, find the points on this graph with $x = 1, 2, 3$, respectively. (If you cannot simplify, just leave it there. E.g. no human brain knows what is sin(1)!)

(iii) For the equation $x^3+y^2 = 5$, find all possible points with $x = 1$ or $x = -2$.

(iv) For the equation $x^2+y^2 = 4$, find all possible points with $x = 0$ or $x = \sqrt{2}$.

Answers: **(i)** $(1, 2)$, $(2, \frac{9}{2})$, $(3, \frac{28}{3})$. **(ii)** $(1, \sin(1)+1)$, $(2, \sin(2)+e^2)$, $(3, \sin(3)+e^6)$. **(iii)** $(1, 2)$, $(1, -2)$, $(-2, \sqrt{13})$, $(-2, -\sqrt{13})$. **(iv)** $(0, 2)$, $(0, -2)$, $(\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$.

Find the slope by x or the given point.

(i) For the function $f(x) = x^2 + \frac{1}{x}$, find the slopes on this graph with $x = 1, 2, 3$, respectively.

(ii) For the function $f(x) = \sin(x) + e^{x^2-x}$, find the slopes on this graph with $x = 1, 2, 3$, respectively.

(iii) For the equation $x^3 + y^2 = 5$, find the slopes at $(1, 2)$ and $(-2, \sqrt{13})$.

(iv) For the equation $x^2 + y^2 = 4$, find the slopes at $(0, 2)$ and $(\sqrt{2}, \sqrt{2})$.

Answers: **(i)** 1, $\frac{15}{4}$, $\frac{53}{9}$. **(ii)** $\cos(1) + 1$, $\cos(2) + 3e^2$, $\cos(3) + 5e^6$. **(iii)** $3x^2 + 2yy' = 0$, you may plug in $(1, 2)$ and $(-2, \sqrt{13})$ and get the slopes y' are $-\frac{3}{4}$ and $-\frac{6}{\sqrt{13}}$. **(iv)** $2x + 2yy' = 0$, you may plug in $(0, 2)$ and $(\sqrt{2}, \sqrt{2})$ and get the slopes y' are 0 and -1.

Find the point by the slope.

(i) For the function $f(x) = x^3 + x + 1$, find the point(s) where the slope is 13.

(ii) For the function $f(x) = e^{x^2-x}$, find the point where the slope is 0.

(iii) For the equation $x^2 + y^2 = 4$, find the point(s) where the slope is 1.

Answers: **(i)** $(2, 11)$, $(-2, -9)$. **(ii)** $(\frac{1}{2}, e^{-\frac{1}{4}})$. **(iii)** $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$.

New description, but same technique! [Parallel means having the same slope. The slope of a line is the coefficient of x.]

(i) For the function $f(x) = x^3 + x + 1$, find its tangent line(s) that is parallel to $y = 13x + 100$. (Meaning solve for $f'(x) = 13$.)

(ii) For the function $f(x) = e^{x^2-x}$, find its tangent line(s) that is parallel to $y = 5$. (Meaning solve for $f'(x) = 0$.)

Answers: **(i)** $y = 13(x-2) + 11$ and $y = 13(x+2) - 9$. **(ii)** $y = e^{-\frac{1}{4}}$.

Indeterminate form. When you evaluate an limit, you will first plug in the given value of x and do the basic examination. Oftenly, you will see it becomes $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$; these are called the *indeterminate form*. For example, $\lim_{x\to 0}\frac{\sin(x)}{x} \to \frac{0}{0}$, even though you know the answer is 1. (Don't write $\lim = \frac{0}{0}$, since the limit is not $\frac{0}{0}$.) Other indeterminate forms include $0\cdot\infty$, $\infty-\infty$, $(1^+)^\infty$, $(0^+)^0$ and $(\infty)^{0^+}$. These forms indicate that two amount are competing each other, while one wishes to go large and the other wishes to go small.

L'Hôpital's Rule. If $\lim_{x\to a}\frac{f(x)}{g(x)}$ becomes an indeterminate form, then the L'Hôpital's Rule says

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

You must check if it is an indeterminate form or not before using the rule, otherwise you will make a mistake like

$$\lim_{x\to 1}\frac{x^2}{x} = \lim_{x\to 1}\frac{2x}{1} = 2.$$

The correct answer is 1 but not 2. If the given question is not in a quotient form, then try to make a quotient form.

Basic type: $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Try the following:

(i) $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+x-6}$ **(iii)** $\lim_{x\to 2}\frac{\sqrt{x+2}-2}{x^2-2x}$ **(v)** $\lim_{x\to 0}\frac{\sin(10x)}{7x}$

(ii) $\lim_{x\to 3}\frac{x^2-4x+3}{x^2-x-6}$ **(iv)** $\lim_{x\to 1}\frac{x^2-2x+1}{3-\sqrt{x+8}}$ **(vi)** $\lim_{x\to 0}\frac{\sin(x^2+2x)}{x}$

Answers: **(i)** 0 **(ii)** $\frac{2}{5}$ **(iii)** $\frac{1}{8}$ **(iv)** 0 **(v)** $\frac{10}{7}$ **(vi)** 2

Other types: $0\cdot\infty$, $\infty-\infty$, $(1^+)^\infty$, or $(\infty)^{0^+}$. If the given form is not a quotient, then try any effort to make a quotient from it and apply the l'Hôpital's rule. If it is of a power form, then take ln first.

(i) $\lim_{x\to\infty} x\sin\frac{1}{x}$ **(iii)** $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ **(v)** $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$ **(vii)** $\lim_{x\to\infty} x^{\frac{1}{x}}$

(ii) $\lim_{x\to 0^+} x\ln x$ **(iv)** $\lim_{x\to\infty} x-\sqrt{x^2+x}$ **(vi)** $\lim_{x\to 1^+} x^{\frac{1}{1-x}}$ **(viii)** $\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$

Answers: **(i)** 1 **(ii)** 0 **(iii)** 0 **(iv)** $-\frac{1}{2}$ **(v)** e **(vi)** e^{-1} **(vii)** 1 **(viii)** 1

Exponential/polynomial growth. For two quantities $f(x) > 0$ and $g(x) > 0$, if we want to know which quantity grows faster when $x\to\infty$, then we can use the test:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \begin{cases} \infty, & \text{then } f \gg g; \\ \text{nonzero finite number}, & \text{then } f\sim g; \\ 0, & \text{then } f \ll g. \end{cases}$$

So the growth of all polynomial is determined by its highest power; e^x is faster than any polynomial, while $\ln(x)$ is slower than any x^r with $r>0$. Try verify $(\ln(x))^2 \ll x \ll e^{0.1x}$.

$$x^0 \ll \ln(x) \ll x^{0.1} \ll x^1 \ll \cdots \ll x^{10} \ll \cdots \ll e^x$$

Antiderivatives and the general antiderivative. Let $f(x)$ be a function. An *antiderivative* is a function $F(x)$ with $F'(x) = f(x)$; the *general antiderivative* is a family of function $F(x) + C$ with $F(x)$ a fixed antiderivative and C any constant. By Mean Value Theorem, the general antiderivative collects all antiderivatives.

Power rule: $x^n \leftarrow \frac{1}{n+1}x^n$. Find the general antiderivatives for the function below.

(i) $2x$ **(iv)** $-3x^{-4}$ **(vii)** $\sqrt{x} + \frac{1}{\sqrt{x}}$ **(x)** $\frac{1}{3}x^{-\frac{2}{3}}$

(ii) x^2 **(v)** $-x^{-3} + x - 1$ **(viii)** $\frac{4}{3}\sqrt[3]{x}$ **(xi)** $\frac{1}{3}x^{-\frac{3}{2}}$

(iii) $x^2 - 2x + 1$ **(vi)** $\frac{3}{2}\sqrt{x}$ **(ix)** $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$ **(xii)** $1 + \frac{4}{3x} - \frac{1}{x^2}$

Answers are: **(i)** x^2+C **(ii)** $\frac{x^3}{3}+C$ **(iii)** $\frac{x^3}{3}-x^2+x+C$ **(iv)** $x^{-3}+C$ **(v)** $\frac{x^{-2}}{2}+\frac{x^2}{2}-x+C$ **(vi)** $x^{\frac{3}{2}}+C$ **(vii)** $\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$ **(viii)** $x^{\frac{4}{3}}+C$ **(ix)** $\frac{3}{4}x^{\frac{4}{3}}+\frac{3}{2}x^{\frac{2}{3}}+C$ **(x)** $x^{\frac{1}{3}}+C$ **(xi)** $-\frac{2}{3}x^{-\frac{1}{2}}+C$ **(xii)** $x + \frac{4}{3}\ln|x| + x^{-1} + C$

Trigonometric functions and exponential functions. Find the general antiderivatives for the function below.

(i) $-\pi\sin(\pi x)$ **(vii)** $4\sec(3x)\tan(3x)$ **(xiii)** 3^{-x}

(ii) $3\sin(x)$ **(viii)** e^x **(xiv)** $\frac{2}{\sqrt{1-x^2}}$

(iii) $\pi\cos(\pi x)$ **(ix)** e^{3x} **(xv)** $\frac{1}{2(x^2+1)}$

(iv) $\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$ **(x)** $e^{\frac{4x}{3}}$ **(xvi)** $\frac{1}{1+4x^2}$

(v) $\sec^2(x)$ **(xi)** 3^x **(xvii)** $x^2 + 2^x$

(vi) $\sec(x)\tan(x)$ **(xii)** 3^{2x} **(xviii)** x^π

Answers are: **(i)** $\cos(\pi x) + C$ **(ii)** $-3\cos(x) + C$ **(iii)** $\sin(\pi x) + C$ **(iv)** $\sin\left(\frac{\pi x}{2}\right) + C$ **(v)** $\tan(x)+C$ **(vi)** $\sec(x)+C$ **(vii)** $\frac{4}{3}\sec(3x)+C$ **(viii)** e^x+C **(ix)** $\frac{1}{3}e^{3x}+C$ **(x)** $\frac{3}{4}e^{\frac{4x}{3}}+C$ **(xi)** $\frac{1}{\ln(3)}3^x+C$ **(xii)** $\frac{1}{2\ln(3)}3^{2x}+C$ **(xiii)** $-\frac{1}{\ln(3)}3^{-x}+C$ **(xiv)** $2\sin^{-1}(x)+C$ **(xv)** $\frac{1}{2}\tan^{-1}(x)+C$ **(xvi)** $\frac{1}{2}\tan^{-1}(2x) + C$ **(xvii)** $\frac{x^3}{3} + \frac{1}{\ln(2)}2^x + C$ **(xviii)** $\frac{1}{\pi+1}x^{\pi+1} + C$

Chain rule and product rule. The following are written of the form of the chain rule or the product rule, can you guess their general antiderivatives?

(i) $4(7x-2)^3 \cdot 7$ **(v)** $(x+1)\cdot 1 + 1\cdot(x+2)$ **(ix)** $\frac{1}{\sin(x)}\cdot\cos(x)$

(ii) $3(x^2+x)^2\cdot(2x+1)$ **(vi)** $\cos(x^2)\cdot(2x)$ **(x)** $e^{x^2}\cdot(2x)$

(iii) $\frac{1}{2}(2x+1)^{-\frac{1}{2}}\cdot 2$ **(vii)** $-\sin(x^2+x)\cdot(2x+1)$ **(xi)** $e^{x^2+x}\cdot(2x+1)$

(iv) $\frac{1}{2}(x^2+x)^{-\frac{1}{2}}\cdot(2x+1)$ **(viii)** $\frac{1}{x^2+x}\cdot(2x+1)$ **(xii)** $x^2\cdot e^x + 2x\cdot e^x$

Answers are: **(i)** $(7x-2)^4 + C$ **(ii)** $(x^2+x)^3 + C$ **(iii)** $\sqrt{2x+1} + C$ **(iv)** $\sqrt{x^2+x} + C$ **(v)** $(x+1)(x+2) + C$ **(vi)** $\sin(x^2) + C$ **(vii)** $\cos(x^2+x) + C$ **(viii)** $\ln(x^2+x) + C$ **(ix)** $\ln(\sin(x)) + C$ **(x)** $e^{x^2} + C$ **(xi)** $e^{x^2+x} + C$ **(xii)** $x^2e^x + C$

Extrema. If the stock prices are described by a function, you want to buy it at the lowest and sell them at the highest; they are the *minimum* and the *maximum* of the function. Two types of extrema are of interests. The *absolute maximum/minimum* is the highest/lowest point over the whole function; while the *local maximum/minimum* is the highest/lowest point over points nearby.

Critical points, boundary points, and absolute extrema. Let $f(x)$ be a function on $[a,b]$. Then a,b are called the *boundary point*, and any point c with $a<c<b$ is called an *interior point*. Whenever a local extrema happens at an interior point c and $f'(c)$ exists, it must be $f'(c)=0$. So to find the extrema, it is enough to check the *critical points* (points c with $f'(c)=0$ or $f'(c)$ DNE) and the boundary points. Find all the critical points and the boundary points, then compare the values of f for these points. Thus, you can find the absolute extrema. Find the absolute extrema for the following functions.

(i) $f(x)=x^2-2x+3, x\in[0,3]$

(ii) $f(x)=x^3-3x^2+2, x\in[1,3]$

(iii) $f(x)=e^x(x^2-3), x\in[-4,2]$

Answers: **(i)** min at $(1,2)$, max at $(3,6)$ **(ii)** min at $(2,-2)$, max at $(3,2)$ **(iii)** min at $(1,-2e)$, max at $(2,e^2)$

Using calculus to draw the graph. In order to find the extrema, the best way is to sketch the graph of the function. The signs of the function and its derivatives provide the information of the graph.

$f>0 \Leftrightarrow$ above x-axis	$f(a)=0 \Leftrightarrow$ crossing x-axis at a	$f<0 \Leftrightarrow$ below x-axis
$f'>0 \Leftrightarrow$ increasing	$f'(a)=0 \Leftrightarrow$ flat ground at a	$f'<0 \Leftrightarrow$ decreasing
$f''>0 \Leftrightarrow$ concave up	$f''(a)=0 \Leftrightarrow$ like a straight line at a	$f''<0 \Leftrightarrow$ concave down

For a function $f(x)$, a point c is called an *inflection point* if $f(x)$ switches its concavity at c; that is, $f''(x)>0$ when $x<c$ and $f''(x)<0$ when $x>c$ (or $f''(x)$ switches from neagtive to positive). If differentiable, an inflection point c must have $f''(c)=0$; but $f''(c)=0$ does not imply c is an inflection point (e.g. $f(x)=x^4$ and $c=0$).

Try to sketch the functions and make it as accurate as possible.

(i) $f(x)=x^3-3x^2+2, x\in\mathbb{R}$ **(ii)** $e^x(x^2-3), x\in\mathbb{R}$

Use `https://www.wolframalpha.com/` and type in `x^3-3x^2+2, -2<x<4` to check your answer (you can try different functions and different domains); also try to spot the critical points, inflection points, and local or absolute extrema.

Absolute extrema for unbounded x. Sometimes the function is not defined on an interval like $[a,b]$. If we suspect $x=c$ is the absolute minimum, then to justify our answer we *need to show* $f'(x)<0$ when $x<c$ and $f'(x)>0$ when $x>c$; for maximum, show $f'(x)>0$ when $x<c$ and $f'(x)<0$ when $x>c$. For questions below, you may first replace y by a formula of x. There answers can be found at the back.

(i) For $x,y>0$, find the maximum of xy when $y=4-x$.

(ii) For $x,y>0$, find the minimum of $2x+8y$ when $xy=9$.

(iii) For $x,y>0$, find the maximum of x^2y when $2x+y=9$.

[ALWAYS remember to justify that your answer is indeed the minimum or the maximum!!!]

Cheating by inequalities!!

Completing a square. This applies only for quadratic polynomial. First write a quadratic polynomial as $f(x) = a(x-b)^2 + c$. When $a > 0$, $f(x)$ attains the minimum when $x = b$; when $a < 0$, $f(x)$ attains the maximum when $x = b$. Be aware of the given domains. Find the absolute extrema of the following functions.

(i) $f(x) = x^2 - 2x + 3, x \in \mathbb{R}$

(ii) $f(x) = -x^2 + 4x, x \in (0, 4)$

(iii) $f(x) = x^2 - 2x + 3, x \in [0, 3]$

(iv) $f(x) = -x^2 + 4x, x \in [3, 5]$

Answers: **(i)** min at $(1, 2)$, no max **(ii)** max at $(2, 4)$, no min **(iii)** min at $(1, 2)$, max at $(3, 6)$ **(iv)** min at $(5, -5)$, max at $(3, 3)$

Arithmetic-Geometric inequality. For n nonnegative numbers $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

where the equality holds if and only if $a_1 = a_2 = \cdots = a_n$. All the following can be solved by replacing y in terms of x. However, try solving them with A-G inequality, which is faster!

(i) For $x, y > 0$, find the minimum of $2x + 8y$ when $xy = 9$.

(ii) For $x, y > 0$, find the minimum of $2x + 3y$ when $x^2 y = 9$.

(iii) For $x, y > 0$, find the maximum of xy when $x + y = 4$.

(iv) For $x, y > 0$, find the maximum of $x^2 y$ when $2x + y = 9$.

Answers: **(i)** min$= 24$ when $x = 6, y = \frac{3}{2}$. **(ii)** min$= 9$ when $x = 3, y = 1$. **(iii)** max$= 4$ when $x = 2, y = 2$. **(iv)** max$= 27$ when $x = 3, y = 3$.

Cauchy inequality. For any $2n$ numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots a_n b_n)^2,$$

were the equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$. You can apply this to a few questions in Calculus I; however, you will see its full power when someone try to bully you by *Lagrange multiplier* in Calculus II.

(i) Find maximum and minimum of $x + y$, provided that $x^2 + y^2 = 8$.

(ii) Find maximum and minimum of $3x + 4y$, provided that $x^2 + y^2 = 4$.

(iii) Find maximum and minimum of $6x + 12y$, provided that $4x^2 + 9y^2 = 1$.

Answers:

(i) max$= 4$ when $x = 2, y = 2$; min$= -4$ when $x = -2, y = -2$.

(ii) max$= 10$ when $x = \frac{6}{5}, y = \frac{8}{5}$; min$= -10$ when $x = -\frac{6}{5}, y = -\frac{8}{5}$.

(iii) max$= 5$ when $x = \frac{3}{10}, y = \frac{4}{15}$; min$= -5$ when $x = -\frac{3}{10}, y = -\frac{4}{15}$.

General guidelines.

1. Setting up many variables as you like.
2. Find out their relations.
3. Think about *what we want* and *what we have.*
4. The derivative means the velocity, or rate of change. *Take the derivatives* on both sides of your equation, and find out the answer.
5. Be aware of the units.

[For questions below, see the pictures at the back.]

Pouring water. A container is of the shape of a cone with height 30 m, bottom as a tip, top as a circle with radius 10 m. A pipe is pouring water into a container with a constant rate. Suppose at some point the water level is of height 6 m.

(i) If the rate of water coming out of the pipe is 12 m^3/min, how fast does the water level go up?

(ii) If the rate of water level going up is 10 m/min, what is the rate of water coming out of the pipe?

You may pick your favorite variables, but here are some suggestions: t as the time, V as water volume, h as the water level (height from the ground), r as the radius of the water surface. You will derive a relation $\frac{\pi}{9}h^2h' = V'$; in particular when $h = 6$, $4\pi h' = V'$.

The answers are: **(i)** $\frac{3}{\pi}$ m/min **(ii)** 40π m/min

Moving cars. Three points A, B, O are on a plane with $\overline{AO} = 3$, $\overline{BO} = 4$ and $OA \perp OB$. Car 1 starts at A driving toward O; Car 2 starts at B driving away from O.

(i) If the speed of Car 1 is 8 and the speed of Car 2 is 6, what is the rate of change of the distance between the two cars?

(ii) If the speed of Car 1 is 1 and the rate of change of the distance between the two cars is 5, what is the speed of Car 2?

Suggested variables: $x = \overline{OA}$, $y = \overline{OB}$, $w = \overline{AB}$. Then you can derive $xx' + yy' = ww'$; when $x = 3$ and $y = 4$, we have $w = 5$ and $3x' + 4y' = 5w'$.

The answers are: **(i)** 0 [Be aware of the direction. Here $x' = -8$ but not 8.] **(ii)** 7

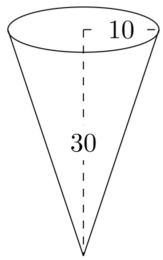
Moving balloons. Two points A and B on the ground are of distance 100 ft. An observer at A is watching a balloon moving upward vertically from B. Let h be the height of the balloon and θ the angle for observer watching the balloon.

(i) If the rate of change of θ is 1 rad/sec at $\theta = \frac{\pi}{3}$ rad, what is the speed of the balloon?

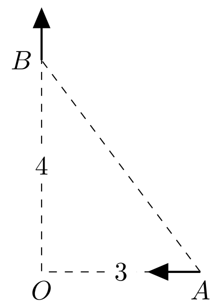
(ii) If the speed of the balloon is 400 ft/sec when $\theta = \frac{\pi}{3}$ rad, what is the rate of change of θ?

All the variables are set up already. You may derive $\sec^2(\theta) \cdot \theta' = \frac{1}{100}h'$; when $\theta = \frac{\pi}{3}$, it becomes $4\theta' = \frac{1}{100}h'$.

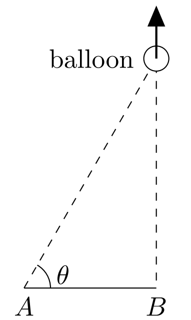
The answers are: **(i)** 400 ft/sec **(ii)** 1 rad/sec



(a) Pouring water



(b) Moving cars



(c) Moving balloons

Area and its boundary. Try the followings:

(i) If a rectangle has one side of length x and its perimeter of length 20, how would you formulate its area in terms of x? What is the legal x? For which x the area is the largest?

(ii) If a rectangle has one side of length x and its area 36, how would you formulate the length of its perimeter in terms of x? What is the legal x? For which x the perimeter is the shortest?

The answers are: **(i)** Area$= x(10 - x)$, $0 < x < 10$, and the area is the largest when $x = 5$. **(ii)** Perimeter$= x + \frac{36}{x}$, $0 < x < 36$, and the perimeter is the shortest when $x = 6$. Beside using the Calculus to solve them, can you do the same by completing the square or the Arithmetic-Geometric inequality?

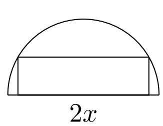
Square inside something. See the pictures at the bottom and try the followings:

(i) The radius of the upper-half circle is 5. How do you formulate the area of the rectangle? When does it become maximum?

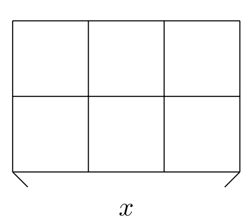
(ii) The length of each 7 lines in the picture sums up as 120. How do you formulate the area of the outside rectangle? When does it become maximum?

(iii) The radius of hemisphere is $\sqrt{3}$ and the radius of the cylinder is x. How do you formulate the volume of the cylinder (don't count the round part at the top)? When does it become maximum?

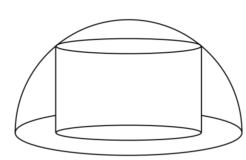
Answers are **(i)** Area$= 2x\sqrt{25 - x^2}$, maximum$= 5$ when $x = \frac{5}{\sqrt{2}}$. **(ii)** Area$= x \cdot \left(\frac{120-3x}{4}\right)$, maximum$= 300$ when $x = 20$. **(iii)** Volume$= \pi x^2\sqrt{3 - x^2}$, maximum$= 2\pi$ when $x = \sqrt{2}$.



(a) **(i)**



(b) **(ii)**



(c) **(iii)**

Definite integral. We know a function can be drawn as a curve on the x, y-plane. Given a function $f(x)$, the *definite integral* $\int_a^b f(x)dx$ is the area of the region between the curve of $f(x)$ and the x-axis with $a \le x \le b$. The idea of the integral aims to sum up infinitely many infinitesimal values. Thinking about driving, the speed of the car is changing all the time, and summing up these values will give you the displacement. Similarly, each vertical line of length $f(x)$ does not increase the area, but summing up all of them gives you the area.

Evaluate the definite integral by antiderivatives. To evaluate an integral $\int_a^b f(x)dx$, you can find an antiderivative $F(x)$, and then the Fundamental Theorem of Calculus tells you

$$\int_a^b f(x)dx = F(b) - F(a).$$

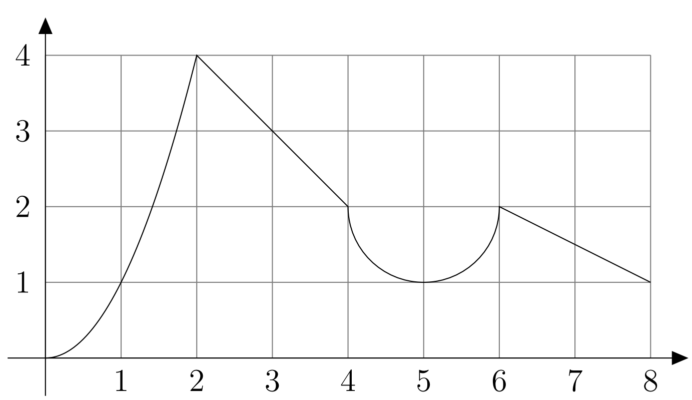
You may use the notation $F(x)\Big|_a^b$ for $F(b) - F(a)$. Try the following integrals:

(i) $\displaystyle\int_0^2 x^2 dx$ **(iii)** $\displaystyle\int_0^{\ln 5} e^{-2x} dx$ **(v)** $\displaystyle\int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}} dx$

(ii) $\displaystyle\int_1^2 \cos(\pi t)dt$ **(iv)** $\displaystyle\int_{-1}^1 \frac{1}{1+x^2} dx$ **(vi)** $\displaystyle\int_e^{e^2} \frac{1}{x} dx$

The answers are: **(i)** $\frac{8}{3}$ **(ii)** 0 **(iii)** $\frac{12}{25}$ **(iv)** $\frac{\pi}{2}$ **(v)** $\frac{\pi}{2}$ **(vi)** 1

Evaluate the definite integral by geometric formulas. As the definite integral is the area of a certain region, we may find out the area easily if the region is some simple geometric shapes, such as rectangles, triangles, disks. Notice that *areas below the x-axis counted as negative*. Let $f(x)$ be the graph drawn below with $f(x) = x^2$ for $0 \le x \le 2$



Evaluate the following integrals:

(i) $\displaystyle\int_0^2 f(x)dx$ **(iii)** $\displaystyle\int_0^4 f(x)dx$ **(v)** $\displaystyle\int_4^5 f(x)dx$

(ii) $\displaystyle\int_2^4 f(x)dx$ **(iv)** $\displaystyle\int_4^6 f(x)dx$ **(vi)** $\displaystyle\int_6^7 f(x)dx$

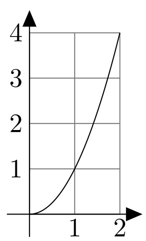
The answers are: **(i)** $\frac{8}{3}$ **(ii)** 6 **(iii)** $\frac{8}{3} + 6$ **(iv)** $4 - \frac{\pi}{2}$ **(v)** $2 - \frac{\pi}{4}$ **(vi)** 1.75

Describing the motion. Recall that the velocity is the instantaneous rate of change of the motion. When $f(t)$ describes the motion of an object, $f'(t)$ and $f''(t)$ describe the velocity and the acceleration. The integration is doing the reverse process of differentiation: The derivative is the instantaneous change, and the integration collects all these instantaneous changes and sums them up. So when you have $v(t)$ as the velocity, then $\int_a^b v(t)dt$ is the displacement between $t = a$ and $t = b$. In particular, $\int_0^b v(t)dt$ is the location of the object at $t = b$, assuming the object is at the origin when $t = 0$. Try the question below.

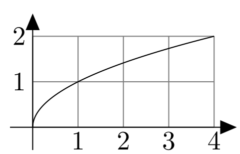
A car starts moving from the origin when $t = 0$ along the x-axis with the velocity $v(t) = t^2 - 2t$. Since $v(t) < 0$ when $0 < t < 2$, you can imagine the car is going left when $0 < t < 2$ and starts going right after $t > 2$. **(i)** Find the location of the car at $t = 2$. **(ii)** Find the location of the car at $t = 3$. **(iii)** Give a formula of the location of the car at time t. **(iv)** When does the car come back to the origin again?

Answers are: **(i)** $-\frac{4}{3}$ **(ii)** 0 **(iii)** $f(t) = \frac{t^3}{3} - t^2$ **(iv)** Solve for $f(t) = 0$ and get $t = 0, 3$. The car come back to the origin when $t = 3$.

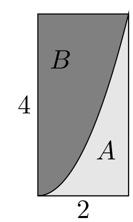
Integral over y-axis. Let's again look at the picture of $f(x) = x^2$ for $0 \leq x \leq 2$. When changing the role of x and y, the function becomes its inverse $g(y) = \sqrt{y}$.



(a) $f(x) = x^2$



(b) $g(y) = \sqrt{y}$



(c) Regions A and B

Find out the value for the following:

(i) $A = \int_0^2 f(x)dx$ **(ii)** $B = \int_0^4 g(y)dy$ **(iii)** $A + B$

Answers are: **(i)** $\frac{8}{3}$ **(ii)** $\frac{16}{3}$ **(iii)** 8

Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus allows us to use the antiderivative for evaluating the definite integral. It states that if $F'(x) = f(x)$ and $f(x)$ is continuous, then $\int_a^b f(t)dt = F(b) - F(a)$. Here a and b can be some variables If you take $b = x$, then we have $G(x) = \int_a^x f(t)dt = F(x) - F(a)$ and so $G'(x) = f(x)$.

Combining chain rules. Finding the antiderivative might not be easy. For example, there is no easy way to write down the antiderivative of $\sin(t^2)$. Thanks to the Fundamental Theorem of Calculus, we can still deal with the derivative of its integral. Let

$$G(x) = \int_3^x \sin(t^2)dt, H(x) = \int_3^{x^2} \sin(t^2)dt, \text{ and } J(x) = \ln(x + H(x)).$$

Find

(i) $\frac{dG(x)}{dx}$ **(ii)** $\frac{dH(x)}{dx}$ **(iii)** $\frac{dJ(x)}{dx}\Big|_{x=\sqrt{3}}$

Answers are: **(i)** $\sin(x^2)$ **(ii)** $\sin(x^4) \cdot (2x)$ **(iii)** $\frac{1+H'(x)}{x+H(x)}\Big|_{x=\sqrt{3}} = \frac{1+\sin(9)\cdot(2\sqrt{3})}{\sqrt{3}}$

Substitution method for indefinite integral. The substitution method says taking integral over $\frac{du}{dx}dx$ is the same as taking integral over du. To be more precise,

$$\int f\frac{du}{dx}dx = \int f du.$$

For example, to find $\int (x^2+1)^5 \cdot 2x dx$, you might think of using $u = x^2+1$, since it looks like a building block of the function. Then you can write it as $\int u^5 \cdot 2xdx$ and see how to interchange du and dx. Since $\frac{du}{dx} = 2x$, we know $2xdx = du$. Thus, the integral becomes $\int u^5 du$, which is $\frac{1}{6}u^6 + C = \frac{1}{6}(x^2+1)^6 + C$. You can check that the derivative of this function is exactly $(x^2+1)^5 \cdot 2x$ and see the substitution method is a reverse work of the chain rule.

For the following function with given u. Use the substitution method to find the answer.

(i) $\int 2(2x+4)^5 dx, u = 2x+4$

(ii) $\int 2x(x^2+5)^{-4} dx, u = x^2+5$

(iii) $\int (3x+2)(3x^2+4x)^4 dx, u = 3x^2+4x$

(iv) $\int x\sin(2x^2)dx, u = 2x^2$

(v) $\int xe^{x^2}dx, u = x^2$

(vi) $\int \frac{2x+1}{x^2+x}dx, u = x^2+x$

Answers are: **(i)** $\frac{1}{6}(2x+4)^6+C$ **(ii)** $-\frac{1}{3}(x^2+5)^{-3}+C$ **(iii)** $\frac{1}{10}(3x^2+4x)^5+C$ **(iv)** $-\frac{1}{4}\cos(2x^2)+C$ **(v)** $\frac{1}{2}e^{x^2}+C$ **(vi)** $\ln|x^2+x|+C$

Finding u by yourself. The key of the substitution method is to find the function u. There is no standard answer for u, it is a good choice once it allows you to find the indefinite integral. This is based on experiences, but there are still some tips for finding u. Usually u is the building block of the given function. For example, you really want to write $(3x^2+4x)^4$ as u^4, $\sin(2x^2)$ as $\sin u$, e^{x^2} as e^u, etc. Another way is to spot the derivative $\frac{du}{dx}$. Most of the time, the given function only have a few factors. Try each of them as $\frac{du}{dx}$ and see if that helps you to find the answer. For example, for $\int \frac{1}{x}\ln x dx$ you may either pick $\frac{du}{dx} = \frac{1}{x}$ or $\frac{du}{dx} = \ln x$, yet $\frac{du}{dx} = \frac{1}{x}$ (so $u = \ln x$) should be the right choice since the integral becomes $\int udu$. (In fact, Calculus I didn't teach you how to find u for $\frac{du}{dx} = \ln x$.) Try the following without the function u given.

(i) $\int \sqrt{3-2s}ds$

(ii) $\int 3y\sqrt{7-3y^2}dy$

(iii) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$

(iv) $\int \sin^2 x\cos x dx$

(v) $\int \tan^2 x\sec^2 x dx$

(vi) $\int (\cos x)e^{\sin x}dx$

(vii) $\int \frac{dx}{x\ln x}$

(viii) $\int \frac{e^x dx}{1+e^x}$

Answers are: **(i)** $-\frac{1}{3}(3-2s)^{\frac{3}{2}}+C$ **(ii)** $-\frac{1}{3}(7-3y^2)^{\frac{3}{2}}+C$ **(iii)** $-2(1+\sqrt{x})^{-1}+C$ **(iv)** $\frac{1}{3}\sin^3 x+C$ **(v)** $\frac{1}{3}\tan^3 x+C$ **(vi)** $e^{\sin x}+C$ **(vii)** $\ln\ln x+C$ **(viii)** $\ln|1+e^x|+C$

Definite integral. While the indefinite integral gives you the antiderivatives, the antiderivatives allow you to evaluate the definite integral. For $\int_a^b f dx$, if your antiderivative is written in terms of u, then you don't need to change u back to x, since you can simply use $u(a)$ and $u(b)$ as the two bounds. Hoever, you can always change u back to x; it just take a few more steps. Try the following.

(i) $\int_{-1}^0 2(2x+4)^5 dx$ **(ii)** $\int_0^{\sqrt{\pi}} x\sin(2x^2)dx$ **(iii)** $\int_0^1 xe^{x^2}dx$

Answers are: **(i)** 672 **(ii)** 0 **(iii)** $\frac{1}{2}(e-1)$

Finding area through integral. If a region is on $a \le x \le b$ and has upper boundary $f(x)$ and lower boundary $g(x)$, then its area is $\int_a^b f(x) - g(x) dx$. Though the formula is easy, one task is to identify which curve is the upper boundary and which curve is the lower one. The best way to do so is draw the graph. Find the area bounded by the given boundaries.

(i) $y = e^{-x}$, $y = x^2$, $x = 1$, $x = 2$. **(ii)** $y = \sin(x) + 3$, $y = \frac{1}{x}$, $x = \pi$, $x = 2\pi$.

Answers are: **(i)** $\frac{7}{3} + e^{-2} - e^{-1}$ **(ii)** $-2 + 3\pi - \ln(2\pi) + \ln(\pi) = -2 + 3\pi - \ln 2$

Finding the intersection. Sometimes the region is defined by two curves only, and you need to find out their intersections by yourself. Find the area enclosed by the two curves provided.

(i) $y = x^2 - 4x - 6$, $y = -x^2 + 4x + 4$. **(ii)** $y = -x^2 - 4x + 5$, $y = 2x - 2$.

Answers are: **(i)** 72 **(ii)** $\frac{256}{3}$

Differentials and linearization. The notation dx is called the differential of x, which means a small amount of change on x. It is basically equal to zero according to its meaning. However, it becomes meaningful when another differential comes into play. For example, $\frac{dy}{dx}$ is the ratio of the small change of y over the small change of x, which is the slope/derivative. The integral $\int_a^b f(x)dx$ is a summation ($\int$) of the area ($f(x)dx$) of rectangles with height ($f(x)$) and base (dx), so it is the area bounded by f and the $y = 0$. Finally, the substitution method or the chain rule says $dy = \frac{dy}{dx}dx$, which means a small change of y is the derivative $\frac{dy}{dx}$ times the corresponding small change of x.

For example, the derivative of $y = e^{2x}$ at $x = 0$ is 2, so we have $dy = 2dx$. We have $y = 1$ at $x = 0$. When x increases by 0.1, y will increase by roughly 0.2, so $y \sim 1.2$ when $x = 0.1$. Formally, *a linearization of $f(x)$ at x_0 is the tangent line $L(x)$ of f at x_0*, When x_1 is nearby x_0, $f(x_1)$ will be very close to $L(x_1)$, which is called the approximation $f(x_1)$.

(i) Find the linearization of $x^{\frac{1}{5}}$ at $x = 32$. **(ii)** Find the approximation of $34^{\frac{1}{5}}$.

Answers are: **(i)** $y = \frac{1}{80}(x - 32) + 2$ **(ii)** $\frac{34-32}{80} + 2 = 2.025$

Separable differential equation. A separable differential equation is of the form of $\frac{dy}{dx} = A(x)B(y)$. You may rewrite it as $\frac{1}{B(y)}dy = A(x)dx$ and take integral on both side with respect to dy and dx, respectively. This will give you an equation describing the relation between x and y. The answer might have an unknown constant involved. When some further condition given, you may solve the C by the condition. Try solving the following equations.

(i) $W' = -\frac{5}{3}tW^{\frac{2}{3}}$. **(ii)** $y' = \frac{\cos(t)}{2y+2}, y(0) = 2$

Answers are: **(i)** $3W^{\frac{1}{3}} = -\frac{5}{6}t^2 + C$ **(ii)** $y^2 + 2y = \sin t + 8$

Half-life. A function that describes the decay of certain energy/particles is usually of the form of $f(t) = Ae^{-kt}$. The half-life time means the length of the time required to make the quantity decrease by half. Mathematically, the half-life time is $t_2 - t_1$ when $f(t_2) = \frac{1}{2}f(t_1)$, which means $Ae^{-kt_2} = \frac{A}{2}e^{-kt_2}$. Dividing by A and Taking natural log function on both sides give you $-kt_2 = -\ln 2 - kt_1$, equivalently $t_2 - t_1 = \frac{\ln 2}{k}$. This value is independent of A and the choice of t_1 and t_2, and that is why scientists choose this value as an indicator of the speed of decay — the decay is slower/faster if the half-life time is longer/shorter.