國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） MATH 103 / GEAI 1215: Linear Algebra I

期末考 January 7, 2019 **Final Examination**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	9 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Find the general solution of the following linear system.

$$\begin{cases} w + 3x + 2y + z = 2 \\ 2w + 6x + 4y + 3z = 9 \\ 3w + 9x + 6y + 3z = 6 \end{cases}$$

That is, find $\vec{\boldsymbol{p}}$ and $\overrightarrow{\boldsymbol{\beta_1}}, \ldots, \overrightarrow{\boldsymbol{\beta_k}}$ such that

$$\{\vec{\boldsymbol{p}} + c_1\overrightarrow{\boldsymbol{\beta_1}} + \cdots + c_k\overrightarrow{\boldsymbol{\beta_k}} : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

2. [5pt] Suppose $S = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ is a set of nonzero vectors in $\mathbb{R}^n$ such that $\vec{\boldsymbol{w}}_i \cdot \vec{\boldsymbol{w}}_j = 0$ for any distinct i and j. (That is, any two vectors in S are orthogonal to each other.) Show that S is linearly independent.

3. [3pt] Let $\vec{\boldsymbol{x}} = \begin{bmatrix} i \\ 2+i \\ 3+2i \end{bmatrix}$ and $\vec{\boldsymbol{y}} = \begin{bmatrix} 3+i \\ 1 \\ 2-i \end{bmatrix}$. Find the values of the inner products $\langle \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} \rangle$, $\langle \vec{\boldsymbol{y}}, \vec{\boldsymbol{x}} \rangle$ and the norm $|\vec{\boldsymbol{x}}|$ in $\mathbb{C}^3$.

4. [2pt] Let $\vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and $\vec{\boldsymbol{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ such that $\vec{\boldsymbol{u}} = \vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}$ with $\vec{\boldsymbol{x}} \in \text{span}\{\vec{\boldsymbol{v}}\}$ and $\langle \vec{\boldsymbol{v}}, \vec{\boldsymbol{y}} \rangle = 0$.

5. [5pt] Let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 6 \\ 3 \end{bmatrix} \right\}$. Find a basis of V and a basis of $V^{\perp}$.

6. [5pt] Consider the following data:

x_i	1	2	3	4	5
y_i	2	2	3	3	4

Find a line $f(x) = ax + b$ such that the error

$$\sum_{i=1}^{5} (f(x_i) - y_i)^2$$

is minimized.

[The orthogonal projection of a vector $\vec{\boldsymbol{v}}$ onto the column space of a matrix $\boldsymbol{A}$ is $\boldsymbol{A}(\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top \vec{\boldsymbol{v}}$. Your answer can be a formula without computing the final answer, but you have to specify all matrices or vectors occurred in your formula.]

7. [5pt] Let $f : V \to W$ be a homomorphism. Show that $f^{-1}(Y)$ is a subspace of V if Y is a subspace of W.

8. [5pt] Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if nullspace$(f) = \{\vec{\mathbf{0}}\}$.

9. Let

$$\vec{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}, \vec{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \vec{\boldsymbol{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \vec{\boldsymbol{u}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a homomorphism such that

$$f(\vec{\boldsymbol{v}}_1) = f(\vec{\boldsymbol{v}}_2) = f(\vec{\boldsymbol{v}}_3) = \vec{\boldsymbol{u}}.$$

(a) [extra 1pt] Find range(f) in set notation and give a brief reason to your answer. [Hint: $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3\}$ is a basis of $\mathbb{R}^3$.]

(b) [extra 2pt] Find the rank of f and the nullity of f.

(c) [extra 2pt] Find a basis of nullspace(f) and give a brief reason to your answer.

10. [extra 2pt] Recall that the trace of a matrix is the sum of its diagonal entries. That is,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{i,i}$$

if $A = \left[a_{i,j}\right]$ is an $n \times n$ matrix. Let $\mathcal{M}_{n\times n}$ be the set of all $n \times n$ matrices. Then

$$\begin{aligned} f : \mathcal{M}_{n\times n} &\to \mathbb{R} \\ A &\mapsto \operatorname{tr}(A) \end{aligned}$$

is a homomorphism. (You don't have to check this fact.) Find the rank of f and the nullity of f in terms of n.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	2	
Total	35 (+7)	