國立中山大學　　NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一）　　**MATH 103 / GEAI 1215: Linear Algebra I**

第一次期中考　　October 8, 2018　　**Midterm 1**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

1. [1pt] Suppose $S = \{\vec{q_1}, \vec{q_2}, \ldots, \vec{q_n}\}$ is a set of n vectors over $\mathbb{R}$. Write down the definition of that "$\vec{v}$ is a *linear combination* of vectors in S."

$$\vec{v} = c_1 \vec{q}_1 + c_2 \vec{q}_2 + \cdots + c_n \vec{q}_n \quad \text{for some } c_1, \ldots, c_n \in \mathbb{R}.$$

2. [1pt] Suppose $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ are two vectors in $\mathbb{R}^n$. Write down the definition of the *inner product* of $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

3. [2pt] Give a linear system of **two equations** in **reduced echelon form** with **four free variables**, and indicate the free variables. [The answer is not unique. You only need to find one.]

$$\begin{aligned} x \quad\quad + z \quad\quad\quad\quad &= 0 \\ y + z + u + v + w &= 0 \end{aligned}$$

free : z, u, v, w.

4. [2pt] Suppose $\boldsymbol{A}$ is a 5×5 nonsingular matrix. What is the minimum number of nonzero entries on $\boldsymbol{A}$? [Justify your answer with an example of such $\boldsymbol{A}$ and explain why the number of nonzero entries cannot be fewer.]

see version A.

5. Let

$$\vec{u} = (1, 0, 1, 0,) \text{ and}$$
$$\vec{v} = \left(-\sqrt{3}, 1, -\sqrt{3}, 1\right).$$

(a) [1pt] Find the length $|\vec{u}|$.

$|\vec{u}| = \sqrt{1^2 + 0^2 + 1^2 + 0^2} = \underline{\underline{\sqrt{2}}}$

(b) [1pt] Find the length $|\vec{v}|$.

$|\vec{v}| = \sqrt{3 + 1 + 3 + 1} = \underline{\underline{\sqrt{8}}}$

(c) [2pt] Find the angle between $\vec{u}$ and $\vec{v}$.

$\vec{u} \cdot \vec{v} = -\sqrt{3} - \sqrt{3} = -2\sqrt{3}$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \dfrac{-2\sqrt{3}}{4} = -\dfrac{\sqrt{3}}{2}$

$\Rightarrow \underline{\underline{\theta = \frac{5}{6}\pi}}$

(d) [2pt] Find a vector $\vec{p}$ that is orthogonal to both of $\vec{u}$ and $\vec{v}$. [The answer is not unique. You only need to find one.]

Let $\vec{p} = (x, y, z, w)$

Solve

$\vec{p} \cdot \vec{u} = x + z = 0$

$\vec{p} \cdot \vec{v} = -\sqrt{3}x + y - \sqrt{3}z + w = 0$.

Any ~~an~~ solution works.

$\underline{\underline{\vec{p} = (0, 1, 0, -1)}}.$

6. [6pt] Find the general solution of the following linear system.

$$\begin{cases} w + 2x + y + 4z = 2 \\ 2w + 4x + 3y - 4z = 3 \\ 3w + 6x + 3y + 12z = \cancel{4}\ 6 \end{cases}$$

That is, find $\vec{p}$ and $\vec{\beta_1}, \ldots, \vec{\beta_k}$ such that

$$\{\vec{p} + c_1\vec{\beta_1} + \cdots + c_k\vec{\beta_k} : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 2 & 4 & 3 & -4 & 3 \\ 3 & 6 & 3 & 12 & \cancel{4}6 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 1 & -12 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 16 & 3 \\ 0 & 0 & 1 & -12 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x, z free

$$\begin{cases} w + 2x + 16z = 3 \\ y - 12z = -1 \end{cases}$$

set $x = c_1$, $z = c_2$

$$\begin{cases} w + 2c_1 + 16c_2 = 3 \\ x = c_1 \\ y - 12c_2 = -1 \\ z = c_2 \end{cases}$$

$$\Rightarrow \begin{cases} w = 3 - 2c_1 - 16c_2 \\ x = c_1 \\ y = -1\ \cancel{+12c_2} + 12c_2 \\ z = c_2 \end{cases}$$

7. [6pt] Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 3 & -3 \\ -2 & -6 & -6 & 7 \\ 3 & 9 & 9 & -9 \end{bmatrix} \text{ and } \boldsymbol{R} = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that $\boldsymbol{R}$ can be obtained from $\boldsymbol{A}$ by performing some row operations. Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{R}$. [The answer is not unique. You only need to find one.]

$$A \xrightarrow[-3\rho_1 + \rho_3]{2\rho_1 + \rho_2} \begin{pmatrix} 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{3\rho_2 + \rho_1} \begin{pmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = R$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} A = R$$

$$\begin{pmatrix} 7 & 3 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} A = R.$$

$$\underline{\underline{C = \begin{pmatrix} 7 & 3 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}}}$$

8. [6pt] Find a matrix $\boldsymbol{A}$ whose reduced echelon form is

$$\begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and every entry of $\boldsymbol{A}$ is nonzero. [The answer is not unique. You only need to find one.]

Do row operations to make every entry nonzero.

$$\begin{pmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_2+\rho_1} \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\rho_1+\rho_3]{2\rho_1+\rho_2} \underline{\underline{\begin{pmatrix} 1 & 3 & 1 & -1 \\ 2 & 6 & 3 & -1 \\ 1 & 3 & 1 & -1 \end{pmatrix}}}$$

A

9. [extra 2pt] There are three types of 2×2 elementary matrices:

(1) $\rho_i \leftrightarrow \rho_j$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(2) $k\rho_i$: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

(3) $k\rho_i + \rho_j$: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Find four matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{E}_4$ of type (2) or type (3) such that

$$\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

[This means the first operation (swapping) can be done by only using the second and the third operations.]

See version A.

[END]

Page	Points	Score
1	6	
2	6	
3	6	
4	6	
5	6	
6	2	
Total	30 (+2)	