國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 November 19, 2018 **Midterm 2**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**35 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [5pt] Find the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 & -2 & -4 \\ -2 & 3 & 3 & 6 \\ 2 & -4 & -3 & -6 \\ 0 & -1 & -1 & -1 \end{bmatrix}.$$

$$(A \mid I) = \left(\begin{array}{cccc|cccc} 1 & -2 & -2 & -4 & 1 & & & \\ -2 & 3 & 3 & 6 & & 1 & & \\ 2 & -4 & -3 & -6 & & & 1 & \\ 0 & -1 & -1 & -1 & & & & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & -2 & -4 & 1 & & & \\ 0 & -1 & -1 & -2 & 2 & 1 & & \\ 0 & 0 & 1 & 2 & -2 & & 1 & \\ 0 & -1 & -1 & -1 & 0 & & & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -2 & -2 & -4 & 1 & & & \\ 0 & -1 & -1 & -2 & 2 & 1 & & \\ 0 & 0 & 1 & 2 & -2 & & 1 & \\ 0 & 0 & 0 & 1 & -2 & -1 & & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & -2 & -4 & 1 & & & \\ & 1 & 1 & 2 & -2 & -1 & & \\ & & 1 & 2 & -2 & & 1 & \\ & & & 1 & -2 & -1 & & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -2 & -2 & 0 & -7 & -4 & 0 & 4 \\ & 1 & 1 & 0 & 2 & 1 & 0 & -2 \\ & & 1 & 0 & 2 & 2 & 1 & -2 \\ & & & 1 & -2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & -3 & 0 & 2 & 0 \\ & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ & & 1 & 0 & 2 & 2 & 1 & -2 \\ & & & 1 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & & & & -3 & -2 & 0 & 0 \\ & 1 & & & 0 & -1 & -1 & 0 \\ & & 1 & & 2 & 2 & 1 & -2 \\ & & & 1 & -2 & -1 & 0 & 1 \end{array}\right), \quad \text{so } A^{-1} = \begin{pmatrix} -3 & -2 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 2 & 2 & 1 & -2 \\ -2 & -1 & 0 & 1 \end{pmatrix}$$

2. [2pt] Suppose V is a vector space over $\mathbb{R}$ and W is a nonempty subset of V. What property (or properties) you have to check in order to make sure W is a subspace of V?

① if $r \in \mathbb{R}$ and $\vec{w} \in W$, then $r\vec{w} \in W$.

② if $\vec{u}, \vec{v} \in W$, then $\vec{u}+\vec{v} \in W$.

3. [3pt] For each of V below, write T or F in the box to indicate V is a vector space over $\mathbb{R}$ or not. If your answer is F, provide a brief reason of why V is not a vector space.

(a) $V = \{\boldsymbol{X} \in \mathcal{M}_{n\times n} : \boldsymbol{AX} = 0\}$. Here $\mathcal{M}_{n\times n}$ is the set of all $n \times n$ real matrices, and $\boldsymbol{A}$ is a matrix in $\mathcal{M}_{n\times n}$.

$\boxed{T}$ Brief reason if F:

(b) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x, y \in \mathbb{Z} \right\}$.

$\boxed{F}$ Brief reason if F: 純量乘法不封閉

(c) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 0 \right\}$.

$\boxed{T}$ Brief reason if F:

(d) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 1 \right\}$.

$\boxed{F}$ Brief reason if F: 加法不封閉, etc. ...

4. [2pt] Let $S = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be a set of vectors in a vector space over $\mathbb{R}$. Write down the definition of that S is linearly independent. (Your answer should be clear in mathematical sense instead of a descriptive sentence in human language.)

If $c_1\vec{w}_1 + \ldots + c_k\vec{w}_k = \vec{0}$,

then $c_1 = \ldots = c_k = 0$.

5. [2pt] Find all possible solutions $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -1 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ -2 & 4 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c_3 is free.

Let $c_3 = 1$. $\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix}$

$$\text{solutions} = \left\{ t\begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

6. [1pt] Is the set $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ linearly independent? Provide your reason.

No. $\frac{3}{2}\begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

7. [5pt] Let

$$V = \operatorname{span}\left(\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}, \begin{bmatrix}1\\3\\3\end{bmatrix}, \begin{bmatrix}4\\-4\\12\end{bmatrix}\right\}\right).$$

Find **a basis** and **the dimension** of V.

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 3 & -4 \\ 3 & 6 & 3 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow & & \uparrow & \end{array}$$

leading variables.

basis $= \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix} \right\rangle$.

dimension $= 2$.

8. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 3 & -4 \\ 3 & 6 & 3 & 12 \end{bmatrix}.$$

(a) [2pt] Find **a basis** and **the dimension** of the row space of $\boldsymbol{A}$.

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 16 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis $= \langle (1, 2, 0, 16), (0, 0, 1, -12) \rangle$

dimension $= 2$.

(b) [3pt] Find **a basis** and **the dimension** of the null space of $\boldsymbol{A}$.

Solve $\begin{pmatrix} 1 & 2 & 0 & 16 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

y, w free.

Let $y = 1, w = 0 \Rightarrow \beta_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Let $y = 0, w = 1 \Rightarrow \beta_2 = \begin{pmatrix} -16 \\ 0 \\ 12 \\ 1 \end{pmatrix}$

basis $= \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -16 \\ 0 \\ 12 \\ 1 \end{pmatrix} \right\rangle$

dimension $= 2$.

9. [5pt] Suppose $S = \{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_k\}$ is a set of nonzero vectors in $\mathbb{R}^n$ such that $\vec{\boldsymbol{w}}_i \cdot \vec{\boldsymbol{w}}_j = 0$ for any distinct i and j. (That is, any two vectors in S are orthogonal to each other.) Show that S is linearly independent.

See version A.

10. [5pt] Let $S = \{\vec{\boldsymbol{w}}_1, \dots, \vec{\boldsymbol{w}}_k\}$ be a set of vectors in a vector space over $\mathbb{R}$. Suppose S is linearly independent. Show that $S \cup \{\vec{\boldsymbol{v}}\}$ is linearly independent if and only if the vector $\vec{\boldsymbol{v}}$ is not in $\mathrm{span}(S)$.

See version A.

11. [extra 2pt] Let $\mathcal{S}_n$ be the set of all $n \times n$ real symmetric matrices. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be two matrices in $\mathcal{S}_3$. Find a basis for

$$V = \{\boldsymbol{X} \in \mathcal{S}_3 : \boldsymbol{AX} = O\}.$$

See version A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	35 (+2)	