國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第一次期中考 October 8, 2018 **Midterm 1**

姓名 Name : ____________________

學號 Student ID # : ____________________

開始前先寫

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

SAMPLE

Do not open this packet until instructed to do so.

發卷後會統一開始

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.

1. [1pt] Suppose $S = \{\vec{z_1}, \vec{z_2}, \ldots, \vec{z_n}\}$ is a set of n vectors over $\mathbb{R}$. Write down the definition of that "$\vec{v}$ is a *linear combination* of vectors in S."

$\vec{v} = c_1 \vec{z}_1 + c_2 \vec{z}_2 + \ldots + c_n \vec{z}_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$.

2. [1pt] Suppose $\vec{p} = (p_1, \ldots, p_n)$ and $\vec{q} = (q_1, \ldots, q_n)$ are two vectors in $\mathbb{R}^n$. Write down the definition of the *inner product* of $\vec{p}$ and $\vec{q}$.

$\vec{p} \cdot \vec{q} = p_1 q_1 + p_2 q_2 + \cdots + p_n q_n$

3. [2pt] Give a linear system in echelon form with two free variables, and indicate the free variables.

$$\begin{aligned} x + y + z + w &= 3 \\ z + w &= 5 \end{aligned}$$

Many possible solutions ...

Be creative.

free: y, w

4. [2pt] Give two 3×3 matrices such that one is singular while the other is nonsingular.

singular

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

nonsingular

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

Many possible solutions ...

Be creative.

5. Let

$$\vec{u} = (0, 1, -1, -1, -1) \text{ and}$$

$$\vec{v} = \left(\sqrt{3}, \frac{2+3\sqrt{3}}{2}, \frac{-2+\sqrt{3}}{2}, \frac{-2+\sqrt{3}}{2}, \frac{-2+\sqrt{3}}{2}\right).$$

(a) [1pt] Find the length $\|\vec{u}\|$.

$\|\vec{u}\| = \sqrt{0^2 + 1^2 + (-1)^2 + (-1)^2 + (-1)^2} = \sqrt{4} = \underline{2}.$

(b) [1pt] Find the length $\|\vec{v}\|$.

$\|\vec{v}\| = \sqrt{3 + \frac{31+12\sqrt{3}}{4} + \frac{7-4\sqrt{3}}{4} + \frac{7-4\sqrt{3}}{4} + \frac{7-4\sqrt{3}}{4}}$

$= \sqrt{16} = \underline{4}.$

(c) [2pt] Find the angle between $\vec{u}$ and $\vec{v}$.

$\vec{u} \cdot \vec{v} = \frac{2+3\sqrt{3}}{2} - \frac{-2+\sqrt{3}}{2} - \frac{-2+\sqrt{3}}{2} - \frac{-2+\sqrt{3}}{2}$

$= 4$

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{4}{2 \cdot 4} = \frac{1}{2}$

$\Rightarrow \theta = \underline{\frac{\pi}{3}}$

6. [2pt] Let $\vec{p} = (1, 2, 3, 4, 5)$. Find a vector $\vec{q}$ that is parallel to $\vec{p}$ and a vector $\vec{r}$ that is orthogonal to $\vec{p}$. [Note: "parallel" means the angle is 0 or π; "orthognal" means the angle is $\frac{\pi}{2}$.]

$\underline{\vec{q} = (2, 4, 6, 8, 10)}$ $\underline{\vec{r} = (-2, 1, 0, 0, 0)}$

Many possible solutions ...

Be creative.

7. [6pt] Find the general solution of the following linear system.

$$\begin{cases} w - x + y - z = -2 \\ 2w - 2x + y + z = 5 \\ 3w - 3x + 2y \phantom{{}+z} = 3 \end{cases}$$

That is, find $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{\beta_1}}, \ldots, \vec{\boldsymbol{\beta_k}}$ such that

$$\{\vec{\boldsymbol{p}} + c_1\vec{\boldsymbol{\beta_1}} + \cdots + c_k\vec{\boldsymbol{\beta_k}} : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

augmented matrix

$$\left(\begin{array}{cccc|c} \multicolumn{4}{c|}{A} & \vec{b} \\ 1 & -1 & 1 & -1 & -2 \\ 2 & -2 & 1 & 1 & 5 \\ 3 & -3 & 2 & 0 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 9 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -2 \\ 0 & 0 & -1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

free

方法1：演算法.

• set $x = z = 0$, solve for $\vec{p}$ with $A\vec{p} = \vec{b}$:

$$\begin{cases} w + y = -2 \\ -y = 9 \end{cases} \Rightarrow \vec{p} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -9 \\ 0 \end{pmatrix}$$

記得代回原式驗算：

$$\begin{cases} 7 - 0 - 9 - 0 = -2 \\ 14 - 0 - 9 + 0 = 5 \\ 21 - 0 - 18 = 3 \end{cases} \checkmark$$

• set $x = 1, z = 0$, solve for $\vec{\beta}_1$ with $A\vec{\beta}_1 = \vec{0}$:

$$\begin{cases} w - 1 + y = 0 \\ -y = 0 \end{cases} \Rightarrow \vec{p} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

驗算：

$$\begin{cases} 1 - 1 = 0 \\ 2 - 2 = 0 \\ 3 - 3 = 0 \end{cases} \checkmark$$

• set $x = 0, z = 1$, solve for $\vec{\beta}_2$ with $A\vec{\beta}_2 = \vec{0}$:

$$\begin{cases} w + y - 1 = 0 \\ -y + 3 = 0 \end{cases} \Rightarrow \vec{p} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

驗算：

$$\begin{cases} -2 + 3 - 1 = 0 \\ -4 + 3 + 1 = 0 \\ -6 + 6 = 0 \end{cases} \checkmark$$

general solution $= \left\{ \begin{pmatrix} 7 \\ 0 \\ -9 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}$

[空間不夠可以用背面]

方法 2：解參數式.

$$\begin{cases} w - x + y - z = -2 \\ \quad -y + 3z = 9 \end{cases}$$

Let $x = c_1$, $z = c_2$.

$$\begin{cases} w - c_1 + y - c_2 = -2 & \Rightarrow w = -2 + c_1 + c_2 - (-9 + 3c_2) \\ & \quad = 7 + c_1 - 2c_2 \\ x = c_1 & \\ -y + 3c_2 = 9 & \Rightarrow y = -9 + 3c_2 \\ z = c_2 & \end{cases}$$

$$\Rightarrow \begin{cases} w = 7 + c_1 - 2c_2 \\ x = c_1 \\ y = -9 + 3c_2 \\ z = c_2 \end{cases}$$

It's also okay to write your answer in this form.

8. [6pt] Find the reduced echelon form of

$$\begin{bmatrix} 1 & 1 & 5 & 2 \\ 1 & 2 & 7 & -3 \\ -2 & -2 & -10 & -3 \end{bmatrix}.$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 5 & 2 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

9. [6pt] Let

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -1 & 2 & -3 & 5 \\ 2 & -4 & 6 & -8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that $\boldsymbol{R}$ can be obtained from $\boldsymbol{A}$ by performing some row operations. Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{R}$.

Record the row operations.

$$A \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{4\rho_2+\rho_1} \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R.$$

Use elementary matrices.

$$\underbrace{\begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{4\rho_2+\rho_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{-2\rho_1+\rho_3} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\rho_1+\rho_2} \cdot A = R$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \cdot A = R.$$

$$\Rightarrow \begin{pmatrix} 5 & 4 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} A = R.$$

$$C = \begin{pmatrix} 5 & 4 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

記得驗算 $CA=R$.

10. [extra 2pt] ???

[END]

Page	Points	Score
1	6	
2	6	
3	6	
4	6	
5	6	
6	2	
Total	30 (+2)	