Sample Solutions for Sample Questions 10

1. Claim: $V^\perp$ is a subspace.

① check: $r \in \mathbb{R}$, $\vec{x} \in V^\perp \Rightarrow r\vec{x} \in V^\perp$.

Let $r \in \mathbb{R}$ and $\vec{x} \in V^\perp$.

Then $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. [defn of $V^\perp$]

This means $(r\vec{x}) \cdot \vec{v} = \cancel{\vec{x} \cdot (r\vec{v})} = r.(\vec{x} \cdot \vec{v}) = 0$ for all $\vec{v} \in V$.

$\Rightarrow r\vec{x} \in V^\perp$.

② check: $\vec{x}, \vec{y} \in V^\perp \Rightarrow \vec{x} + \vec{y} \in V^\perp$.

Let $\vec{x}, \vec{y} \in V^\perp$.

Then $\vec{x} \cdot \vec{v} = 0 = \vec{y} \cdot \vec{v}$ for all $\vec{v} \in V$ [defn of $V^\perp$]

So

$\vec{v} \cdot (\vec{x} + \vec{y}) = \vec{v} \cdot \vec{x} + \vec{v} \cdot \vec{y} = 0 + 0 = 0$ for all $\vec{v} \in V$

$\Rightarrow \vec{x} + \vec{y} \in V^\perp$.

2. Claim: $(V^{\perp})^{\perp} = V$

~~Suppose $\vec{x} \in (V^{\perp})^{\perp}$.~~

~~Then $\vec{x} \cdot \vec{w} = 0$ for all $\vec{w} \in V^{\perp}$. [defn of $(V^{\perp})^{\perp}$]~~

First we ~~claim~~ show that $(V^{\perp})^{\perp} \supseteq V$.

Suppose $\vec{x} \in V$.

Then for any $\vec{w} \in V^{\perp}$, $\vec{w} \cdot \vec{x} = 0$.

This means $\vec{x} \in (V^{\perp})^{\perp}$.

$\Rightarrow (V^{\perp})^{\perp} \supseteq V$.

Now assume $r = \dim V$ and V is a subspace in an n-dimensional space.

Then
$$r = \dim V$$
$$n - r = \dim V^{\perp}$$
$$r = n - (n-r) = \dim (V^{\perp})^{\perp}.$$

Thus $(V^{\perp})^{\perp} \supseteq V$ with $\dim (V^{\perp})^{\perp} = \dim V$.

This means $(V^{\perp})^{\perp} = V$.

如果 $\vec{y} \in (V^{\perp})^{\perp}$ 但 $\vec{y} \notin V$.
令 $\langle \vec{v}_1, \ldots, \vec{v}_r \rangle$ 為 V 的一組 basis.
那 $\langle \vec{v}_1, \ldots, \vec{v}_r, \vec{y} \rangle$ 就是 $(V^{\perp})^{\perp}$ 中的一個獨立集且個數 $r+1 >$ 維度 r.

3. Claim: $V^\perp \cap V = \{\vec{0}\}$.

"$\supseteq$" Since $\vec{0} \in V^\perp$, $\vec{0} \in V$,

we know $\vec{0} \in V^\perp \cap V$.

That is $V^\perp \cap V \supseteq \{\vec{0}\}$.

"$\subseteq$" Suppose $\vec{x} \in V^\perp \cap V$.

Then $\vec{x} \in V^\perp$ and $\vec{x} \in V$.

By definition of $V^\perp$,

$$\underset{\substack{\cap \\ V^\perp}}{\vec{x}} \cdot \underset{\substack{\cap \\ V}}{\vec{x}} = 0 \Rightarrow |\vec{x}|^2 = 0.$$

Thus, $\vec{x} = \vec{0}$.

This means $V^\perp \cap V \subseteq \{\vec{0}\}$.

4. Let $A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 1 & 1 \\ 3 & -3 & 2 & 0 \end{pmatrix}$.

Then $V = \text{Rowspace}(A)$.

and $V^{\perp} = \text{Nullspace}(A)$.

Solve
$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 1 & 1 \\ 3 & -3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 3 & -3 & 2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

y, w free.

Let $y = 1, w = 0 \Rightarrow \vec{\beta}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Let $y = 0, w = 1 \Rightarrow \vec{\beta}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$.

$\Rightarrow \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ is a basis of $V^{\perp}$.

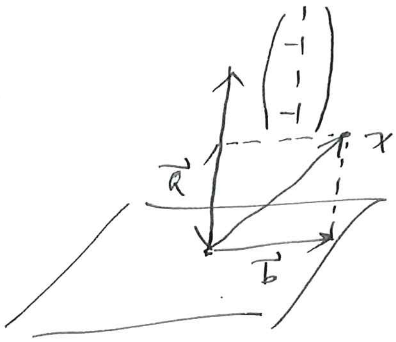
5.

$$\cos\theta = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|\cdot|\vec{y}|} = \frac{-2}{\sqrt{30}\cdot\sqrt{4}} = \frac{-1}{\sqrt{30}}$$

$$\vec{x}\cdot\vec{y} = 1-2+3-4 = -2$$

$$|\vec{x}|^2 = 1^2+2^2+3^2+4^2 = 30$$

$$|\vec{y}|^2 = 1^2+(-1)^2+1^2+(-1)^2 = 4$$



$x-y+z-w=0$

$\vec{x} = \vec{a}+\vec{b}$

$\vec{a}$ = projection of $\vec{x}$ onto $\vec{y}$

$$= |\vec{x}|\cdot\cos\theta\cdot\frac{\vec{y}}{|y|}$$

$$= \sqrt{30}\cdot\frac{-2}{\sqrt{30}\cdot\sqrt{4}}\cdot\frac{1}{\sqrt{4}}\cdot\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}.$$

$\vec{b}$ = projection of $\vec{x}$ onto $x-y+z-w=0$

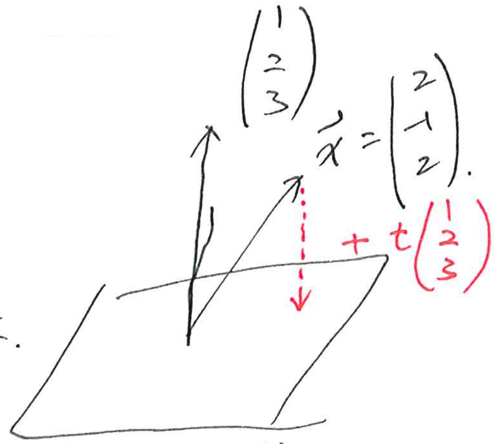
$$= \cancel{\vec{y}\,\vec{a}}\ \vec{x}-\vec{a}$$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}1.5\\1.5\\3.5\\3.5\end{pmatrix}$$

6.

高中~~作~~做法：

法向量： $\begin{pmatrix}1\\2\\3\end{pmatrix}$.



$V = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x+2y+3z=0\right\}$.

找一個 t 讓

$\begin{pmatrix}2\\-1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}$ 落在 $x+2y+3z=0$ 上.

$$\begin{pmatrix}2\\-1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2+t\\-1+2t\\2+3t\end{pmatrix}$$

$\Rightarrow (2+t)+2(-1+2t)+3(2+3t)=0.$

$\Rightarrow 6+14t=0 \Rightarrow t=-\frac{6}{14}=-\frac{3}{7}.$

所以 projection $= \begin{pmatrix}2\\-1\\2\end{pmatrix} - \frac{3}{7}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}11/7\\-13/7\\5/7\end{pmatrix}.$

矩陣投影：

Let $A = \begin{pmatrix}2 & 3\\-1 & 0\\0 & -1\end{pmatrix}$

$\Rightarrow$ projection $= A(A^TA)^{-1}A^T\vec{x}$

$$= \begin{pmatrix}11/7\\-13/7\\5/7\end{pmatrix}$$

7. $V = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 2 \\ 0 \end{pmatrix} \right\}$

$\longrightarrow$ basis $= \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

參考期中考題

找 basis.

$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

一定要用 basis

不然 A^TA 會沒有 inverse.

※ 不能用 $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$ ✗

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow$ projection $= A(A^TA)^{-1}A^T\vec{x}$

$= \begin{pmatrix} 2/3 \\ -2/3 \\ -1/2 \\ 17/6 \end{pmatrix}$