Sample Solutions for Sample Questions 13.

1. (a) $f : R^3 \longrightarrow R^3$ is linear.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

$$f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = f\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ x_1+x_2 \\ x_1+x_2 \end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_2 \\ x_2 \end{pmatrix}$$

same.

$$f\left(r\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) = f\begin{pmatrix} rx_1 \\ ry_1 \\ rz_1 \end{pmatrix} = \begin{pmatrix} rx_1 \\ rx_1 \\ rx_1 \end{pmatrix}$$

$$r \cdot f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = r\begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix}$$

same.

(b) Yes, an isomorphism is a homomorphism.

(c) No, $f : R \rightarrow R$, $x \longmapsto x^3$ is not a homomorphism.

$$f(1+1) = f(2) = 8$$
$$f(1)+f(1) = 1+1 = 2$$

not the same.

2.

[Key: Write $\begin{bmatrix} 9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Then $f\begin{pmatrix} 9 \\ 8 \end{pmatrix} = c_1(1+x+x^2) + c_2(-2+3x+x^2)$]

Solve $\begin{bmatrix} 9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\Rightarrow \begin{cases} c_1 + 3c_2 = 9 \\ 2c_1 + c_2 = 8 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}$$

So $f\begin{pmatrix} 9 \\ 8 \end{pmatrix} = f\left(3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$

$$= 3 \cdot f\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \cdot f\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= 3(1+x+x^2) + 2(-2+3x+x^2).$$

$$= -1 + 9x + 5x^2.$$

3.

Assumption : $\begin{cases} B = \{\vec{v}_1, \dots, \vec{v}_k\} \text{ is linearly indep.} \\ f : V \longrightarrow W \text{ is a homomorphism.} \\ f \text{ is one-to-one} \end{cases}$

[Method: Suppose $c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k) = 0$.
Show $c_1 = \dots = c_k = 0$.]

Suppose $c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k) = 0$

Since f is a homomorphism,

$$f(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) = \vec{0}.$$

Since f is one-to-one and $f(\vec{0}) = \vec{0}$,

it must be $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$.

Since B is indep.,

$c_1 = \dots = c_k = 0$.

Goal: $f(B)$ is indep.

4.

Assumption: $\begin{cases} \text{span}(B) = V, & B = \{\vec{v}_1, \ldots, \vec{v}_k\}. \\ f: V \to W \text{ is a homomorphism} \\ f \text{ is onto.} \end{cases}$

[Method: For each $\vec{w} \in W$, find $c_1, \ldots, c_k$ such that $c_1 \cdot f(\vec{v}_1) + \ldots + c_k \cdot f(\vec{v}_k) = \vec{w}$.]

Suppose $\vec{w} \in W$

Since f is onto,

there exist $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$.

Since $\text{span}(B) = V$, there are $c_1, \ldots, c_k$

such that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{v}$.

Thus,

$$f(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = f(\vec{v}) = \vec{w}.$$

Since f is a homomorphism,

$$\vec{w} = f(c_1\vec{v}_1 + \ldots + c_k\vec{v}_k) = c_1 f(\vec{v}_1) + \ldots + c_k f(\vec{v}_k)$$

Goal: $\text{span}(f(B)) = W$.

5.

$$T : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$$

$$a+bx+cx^2+dx^3 \longmapsto b+2cx+3dx^2$$

$$\text{range}(f) = \{ b+2cx+3dx^2 \mid a,b,c,d \in \mathbb{R} \}$$

= 全部的二次(含以下)多項式.

$$= \mathcal{P}_2.$$

$$\text{rank} = \dim \text{range}(f) = \dim \mathcal{P}_2 = 3.$$

$$\text{null space}(f) = \{ a+bx+cx^2+dx^3 \mid b+2cx+3dx^3=0,\ a,b,c,d \in \mathbb{R} \}$$

$$= \{ a \mid a \in \mathbb{R} \}.$$

$$= \mathcal{P}_0.$$

$$\text{nullity} = \dim \text{null space}(f) = \dim \mathcal{P}_0 = 1.$$

For 6, 7,

See the proof in the lecture notes.

Pretending you are taking the exam,

practice how to write it down.