Sample Solutions for Sample Questions 14.

1. "$\Rightarrow$" Assumption $\begin{cases} f \text{ is homomorphism} \\ f \text{ is one-to-one} \end{cases}$.

Since f is one-to-one,

$$f(\vec{x}) = f(\vec{y}) \Rightarrow \vec{x} = \vec{y}.$$

Since f is homomorphism,

$$f(\vec{0}_V) = \vec{0}_W.$$

Thus, $f(\vec{x}) = \vec{0}_W = f(\vec{0}_V) \Rightarrow \vec{x} = \vec{0}_V$.

Goal: $\text{nullspace}(f) = \{\vec{0}\}$.

"$\Leftarrow$" Assumption $\begin{cases} f \text{ is homomorphism} \\ \cancel{f \text{ has nullspace}(\{\vec{0}\}} \\ \text{nullspace}(f) = \{\vec{0}\}. \end{cases}$

Method: Show $f(\vec{x}) = f(\vec{y}) \Rightarrow \vec{x} = \vec{y}$.

Suppose $f(\vec{x}) = f(\vec{y})$. Then $\underbrace{f(\vec{x}-\vec{y}) = f(\vec{x}) - f(\vec{y})}_{\because f \text{ is homomorphism}} = \vec{0}$.

Since $\text{nullspace}(f) = \{\vec{0}\}$ and $f(\vec{x}-\vec{y}) = \vec{0}$,

we know $\vec{x} - \vec{y} = \vec{0}$.

That is, $\vec{x} = \vec{y}$.

Goal: f is one-to-one.

2. "homomorphism"

~~$f(c_1\vec{w}_1 + c_2\vec{w}_2$~~

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$.

Then $f(c_1\vec{w}_1 + c_2\vec{w}_2) = A(c_1\vec{w}_1 + c_2\vec{w}_2)$

$$= A(c_1\vec{w}_1) + A(c_2\vec{w}_2)$$

$$= c_1 A\vec{w}_1 + c_2 A\vec{w}_2$$

$$= c_1 f(\vec{w}_1) + c_2 f(\vec{w}_2).$$

domain $= \mathbb{R}^n$

codomain $= \mathbb{R}^m$

range $(f) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Colspace}(A)$.

nullspace $(f) = \{\vec{x} \mid A\vec{x} = \vec{0}\} = \text{nullspace}(A)$.

3. range(f) = Colspace(A).

$$A \longrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} \uparrow & & \uparrow & \\ & \text{leading}. & & \end{array}$$

So $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$ is a basis of range(f).

$\Rightarrow$ rank $=2$.

nullspace(f) = nullspace(A).

Solve $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{array}{cccc} & \uparrow & & \uparrow \end{array}$$

y, w free

Let $y=1$, $w=0$. $\Rightarrow \vec{\beta}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Let $y=0$, $w=1$ $\Rightarrow \vec{\beta}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix}$.

So $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$ is a basis of nullspace(f).

$\Rightarrow$ ~~ra~~ nullity $=2$.

Check: rank + nullity $= 2+2=4$. $=$ # of columns.

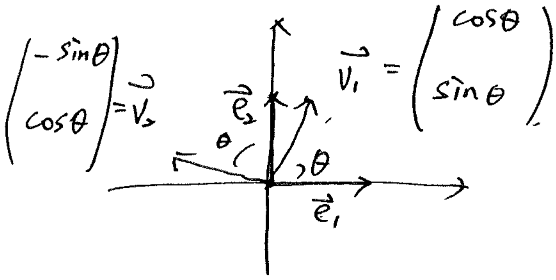
4. If $A = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$,

then $$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \vec{e}_1 &\longrightarrow \vec{v}_1 \\ \vec{e}_2 &\longrightarrow \vec{v}_2 . \end{aligned}$$

Now $f(e_1) = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$, $f(e_2) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$.

So $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$.

5. f : rotation



$$f(\vec{e}_1) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$f(\vec{e}_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

6.

Claim: f is one-to-one $\Leftrightarrow$ $\overset{m\times n}{A}$ has full column rank.

"$\Rightarrow$" f is one-to-one

$\Rightarrow$ nullspace $(f) = \{\vec{0}\}$. [Problem 1]

$\Rightarrow$ nullity $= 0$ and rank $= n$ [dimension theorem]

$\Rightarrow$ ~~dim 6~~ dim range(f) = dim Colspace(A) $= n$

$\Rightarrow$ A has full column rank.

"$\Leftarrow$" A has full column rank

$\Rightarrow$ dim Colspace $(A) = n$

$\Rightarrow$ rank = dim range(f) $= n$

$\Rightarrow$ nullity $= 0$ [dimension theorem]

$\Rightarrow$ f is one-to-one [Problem 1.]

7. Claim: f is onto $\Leftrightarrow$ A has full row rank.

"$\Rightarrow$" f is onto

$\Rightarrow$ $\text{range}(f) = \mathbb{R}^m$

$\Rightarrow$ $\cancel{\text{rank} =}$ $\dim \text{Colspace}(A) = \dim \text{range}(f) = m$

$\Rightarrow$ $\dim \text{Rowspace}(A) = \dim \text{Colspace}(A) = m$

$\Rightarrow$ A has full row rank.

"$\Leftarrow$" f has full row rank

$\Rightarrow$ $\dim \text{Rowspace}(A) = m$.

$\Rightarrow$ $\dim \text{Colspace}(A) = m$

$\Rightarrow$ $\dim \text{range}(f) = m$ ~~and~~

$\Rightarrow$ since $\text{range}(f) \subseteq \mathbb{R}^m$ and $\dim \text{range}(f) = m$,

$\text{range}(f) = \mathbb{R}^m$

$\Rightarrow$ f is onto.