Sample solutions for Sample Questions 1

1. (a)

$$\begin{aligned} x + y &= 5 \\ x - 4y &= 10 \end{aligned} \xrightarrow{-\rho_1+\rho_2} \begin{aligned} x + y &= 5 \\ -5y &= 5 \end{aligned}$$

$\Rightarrow$ unique solution

(b)

$$\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned} \xrightarrow{-3\rho_1+\rho_2} \begin{aligned} -x - y &= 1 \\ 0 &= -1 \end{aligned}$$

$\Rightarrow$ no solution

(c)

$$\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ 2x + 2y + 2z &= 20 \\ 2x - 6y &= -20 \end{aligned} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{aligned} 2x - 2y + z &= 0 \\ 4y + z &= 20 \\ 2x + 2y + 2z &= 20 \\ 2x - 6y &= -20 \end{aligned}$$

$$\xrightarrow[-\rho_1+\rho_4]{-\rho_1+\rho_3} \begin{aligned} 2x - 2y + z &= 0 \\ 4y + z &= 20 \\ 4y + z &= 20 \\ -4y - z &= -20 \end{aligned} \xrightarrow[\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{aligned} 2x - 2y + z &= 0 \\ 4y + z &= 20 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$\Rightarrow$ many solutions

2.

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned} \xrightarrow{-3\rho_1+\rho_2} \begin{aligned} x - y &= 1 \\ 0 &= k-3 \end{aligned}$$

so $k-3=0 \Rightarrow k=3$.

3.

$$\begin{aligned} x + 2y + 3z &= 10 \\ 2x - 2y + z &= 5 \\ x + 8y + 8z &= k \end{aligned} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x + 2y + 3z &= 10 \\ -6y - 5z &= -15 \\ 6y + 5z &= k-10 \end{aligned}$$

$$\xrightarrow{\rho_2+\rho_3} \begin{aligned} x + 2y + 3z &= 10 \\ -6y - 5z &= -15 \\ 0 &= k - 10 - 15 \end{aligned}$$

$\Rightarrow k = 25$.

4.

$$\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + \cancel{75}7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned} \xrightarrow[-2\rho_1+\rho_4]{\substack{-3\rho_1+\rho_2 \\ -\rho_1+\rho_3}} \begin{aligned} x - 3y &= b_1 \\ 10y &= b_2 - 3b_1 \\ 10y &= b_3 - b_1 \\ 10y &= b_4 - 2b_1 \end{aligned}$$

$$\xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{aligned} x - 3y &= b_1 \\ 10y &= b_2 - 3b_1 \\ 0 &= (b_3 - b_1) - (b_2 - 3b_1) \\ 0 &= (b_4 - 2b_1) - (b_2 - 3b_1) \end{aligned}$$

$$\Rightarrow \begin{cases} 2b_1 - b_2 + b_3 = 0 \\ b_1 - b_2 + b_4 = 0 \end{cases}$$

5. $f(1) = a + b + c = 2$

$f(-1) = a - b + c = 6 \xrightarrow[-4\rho_1+\rho_3]{-\rho_1+\rho_2}$

$f(2) = 4a + 2b + c = 3$

$a + b + c = 2$

$-2b = 4$

$-2b - 3c = -5$

$\xrightarrow{-\rho_2+\rho_3}$

$a + b + c = 2$

$-2b = 4$

$-3c = -9$

$\Rightarrow$

$a = 1$

$b = -2$

$c = 3$

6.

$x + y = 5 \quad [\times a]$

$+) \quad 4x - y = 6 \quad [\times b]$

$11x + y = 27$

Compare each column.

$a + 4b = 11$

$a - b = 1$

$5a + 6b = 27$

$\xrightarrow[-5\rho_1+\rho_3]{-\rho_1+\rho_2}$

$a + 4b = 11$

$-5b = -10$

$-14b = -28$

$\Rightarrow$

$a = 3$

$b = 2$

7. Suppose the four integers are x, y, z, w.

According to the assumption,

$$\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + w = 29$$

$$\frac{1}{3}x + \frac{1}{3}y + z + \frac{1}{3}w = 23$$

$$\frac{1}{3}x + y + \frac{1}{3}z + \frac{1}{3}w = 21$$

$$+)\quad x + \frac{1}{3}y + \frac{1}{3}z + \frac{1}{3}w = 17$$

$$2x + 2y + 2z + 2w = 90$$

$$\Rightarrow \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + \frac{1}{3}w = 15.$$

$$\frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + w = 29$$

$$-)\quad \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z + \frac{1}{3}w = 15$$

$$\frac{2}{3}w = 14 \Rightarrow w = 21.$$

Similarly, $z = \frac{3}{2}(23-15) = 12$

$y = \frac{3}{2}(21-15) = 9$

$x = \frac{3}{2}(17-15) = 3$