Sample solutions for Sample Questions 2

1. Consider the three types of row operations.

① $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{\rho_i \leftrightarrow \rho_j} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ change the positions only.

② $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{k\rho_i} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← replace 0 by $k \cdot 0 = 0$.

③ $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{k\rho_i + \rho_j} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← replace 0 by $k \cdot 0 + 0 = 0$.

So $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow[\text{row operations}]{\text{any}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

2.

Echelon form:

$$\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} \overset{\text{leading}}{3} & 6 & 18 \\ 0 & \underset{\text{free}}{0} & 0 \end{array}\right).$$

particular solution:

set $y=0$.

$3x \quad = 18 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$

homogeneous solution:

set $y=1$, solve $\left(\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$3x + 6 = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So general solution = particular + homogeneous

$$= \left\{ \begin{pmatrix} 6 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\}.$$

3. Echelon form

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 1 & 0 & 4 \\ 1 & 1 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 & 3 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

leading　　free

particular :

let $y = z = 0$.

$$\begin{aligned} w + x &= 1 \\ x &= 3 \end{aligned} \Rightarrow \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

homogeneous : solve $\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

① let $y = 1, z = 0$

$$\begin{aligned} w + x - 1 &= 0 \\ x + 2 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

② let $y = 0, z = 1$

$$\begin{aligned} w + x + 1 &= 0 \\ x - 1 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

So general = particular + homogeneous

$$= \left\{ \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

4. Echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{ccccc} u & w & x & y & z \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

leading

free.

particular: set $w=x=y=z=0$.

$u=1 \Rightarrow \begin{pmatrix} u \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

homogeneous: solve $u+w+x+y+z=0$.

① let $w=1$, $x=y=z=0$.

~~u = 1~~ $u+1=0 \Rightarrow \begin{pmatrix} u \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

② let $x=1$, $w=y=z=0$

$u+1=0 \Rightarrow \begin{pmatrix} u \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

③ let $y=1$, $w=x=z=0$

$u+1=0 \Rightarrow \begin{pmatrix} u \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

④ let $z=1$, $w=x=y=0$

$u+1=0 \Rightarrow \begin{pmatrix} u \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ general solution $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : a,b,c,d \in \mathbb{R} \right\}$

5. Do the echelon form.

(a)
$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-\rho_1+\rho_4]{-\rho_1+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}\rho_3+\rho_4} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 0 \\ & & -2 & 1 & 0 \\ & & & -1.5 & 0 \end{array}\right)$$

$\Rightarrow$ unique solution

$\Rightarrow$ nonsingular.

[Note: So keep writing $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is redundant.].

(b)
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix} \xrightarrow[\substack{-\rho_2+\rho_3 \\ -\rho_1+\rho_2}]{-\rho_3+\rho_4} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$$

$$\xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

free variables.

So there are many solutions

$\Rightarrow$ singular.

6. Solve $a\begin{pmatrix}1\\4\end{pmatrix}+b\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$.

That is $\begin{pmatrix}1&1\\4&5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$.

$$\left(\begin{array}{cc|c}1&1&2\\4&5&3\end{array}\right)\xrightarrow{-4\rho_1+\rho_2}\left(\begin{array}{cc|c}1&1&2\\0&1&-5\end{array}\right)\Rightarrow\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}7\\-5\end{pmatrix}$$

Check $7\begin{pmatrix}1\\4\end{pmatrix}-5\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$.

7. Solve $\begin{pmatrix}2&3\\1&0\\0&0\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$.

From 2nd row $\Rightarrow$ $a=0$.

$\Rightarrow$ solve $b\begin{pmatrix}3\\0\\0\\2\end{pmatrix}=\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$ $\Rightarrow$ impossible.