Sample Solutions for Sample Questions 4

1.

$$\begin{array}{c} \quad A \qquad\qquad I_4 \\ \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & & & \\ 1 & 2 & -4 & 2 & & 1 & & \\ 2 & 3 & -6 & 5 & & & 1 & \\ 3 & 3 & -9 & 4 & & & & 1 \end{array}\right) \end{array} \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & & & \\ 0 & 1 & -1 & 1 & -1 & 1 & & \\ 0 & 1 & 0 & 3 & -2 & & 1 & \\ 0 & 0 & 0 & 1 & -3 & & & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & & & \\ 0 & 1 & -1 & 1 & -1 & 1 & & \\ 0 & 0 & 1 & 2 & -1 & -1 & 1 & \\ 0 & 0 & 0 & 1 & -3 & & & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 0 & 4 & & & -1 \\ 0 & 1 & -1 & 0 & 2 & 1 & & -1 \\ 0 & 0 & 1 & 0 & 5 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & & & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 19 & -3 & 3 & -7 \\ 0 & 1 & 0 & 0 & 7 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 5 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & & & 1 \end{array}\right) \longrightarrow \begin{array}{c} \left(\begin{array}{cccc|cccc} 1 & & & & 12 & -3 & 2 & -4 \\ & 1 & & & 7 & 0 & 1 & -3 \\ & & 1 & & 5 & -1 & 1 & -2 \\ & & & 1 & -3 & 0 & 0 & 1 \end{array}\right) \\ \qquad I_4 \qquad\qquad\qquad B. \end{array}$$

Check:

$$\begin{pmatrix} 12 & -3 & 2 & -4 \\ 7 & 0 & 1 & -3 \\ 5 & -1 & 1 & -2 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & -4 & 2 \\ 2 & 3 & -6 & 5 \\ 3 & 3 & -9 & 4 \end{pmatrix} = I_4 .$$

$$\begin{pmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & -4 & 2 \\ 2 & 3 & -6 & 5 \\ 3 & 3 & -9 & 4 \end{pmatrix} \begin{pmatrix} 12 & -3 & 2 & -4 \\ 7 & 0 & 1 & -3 \\ 5 & -1 & 1 & -2 \\ -3 & 0 & 0 & 1 \end{pmatrix} = I_4 .$$

2. (a)

$$0 = 0 + 0x + 0x^2 + 0x^3.$$

verify:

$$(0 + 0x + 0x^2 + 0x^3) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$

$$= (0 + a_0) + (0 + a_1) x + (0 + a_2) x^2 + (0 + a_3) x^3$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

(b) zero matrix $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

verify:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

$$= \begin{pmatrix} 0 + a_{11} & 0 + a_{12} & 0 + a_{13} & 0 + a_{14} \\ 0 + a_{21} & 0 + a_{22} & 0 + a_{23} & 0 + a_{24} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

(c) zero $= f : [0,1] \to \mathbb{R}.$
$x \mapsto 0.$

verify:

$(f+g)(x) = f(x) + g(x)$

$= g(x)$ for all $x \in [0,1]$. $\Rightarrow f + g = g$

(d) zero $= f : \mathbb{N} \to \mathbb{R}.$
$x \mapsto 0.$

verify:

$(f+g)(x) = f(x) + g(x)$

$= g(x)$ for all $x \in \mathbb{N}$. $\Rightarrow f + g = g$

3. (a) $v = -3 - 2x + x^2$

additive inverse $w = 3 + 2x - x^2$.

verify:

$$(-3 - 2x + x^2) + (3 + 2x - x^2)$$

$$= (3+3) + (-2+2)\,x + (1-1)x^2$$

$$= 0 + 0x + 0x^2 + 0x^3$$ ← zero in the space.

(b). $v = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$

additive inverse $w = \cancel{\begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}}. \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$

verify:

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} (1-1) & (-1+1) \\ (0+0) & (3-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

↑ zero in the space.

(c). $v = 3e^x - 2e^{-x}$

additive inverse $w = -3e^x + 2e^{-x}$.

verify:

$$3e^x - 2e^{-x} - 3e^x + 2e^{-x} = 0$$ ← zero in the space.

4. ① if $\vec{b} = 0$, then V is a vector space.

Goal: ② if $\vec{b} \neq 0$, then V is not a vector space.

① Verify all 10 properties:

Suppose $\vec{x}, \vec{y}, \vec{z} \in V$ and $r, s \in \mathbb{R}$.

(1) $A\vec{x} = \vec{0}$, $A\vec{y} = \vec{0}$

$\Rightarrow A(\vec{x}+\vec{y}) = \vec{0}$, so $\vec{x}+\vec{y} \in V$.

(2) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ in $\mathbb{R}^n$.

(3) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ in $\mathbb{R}^n$.

(4) $\vec{0} \in V$ because $A\vec{0} = \vec{0}$.

also, $\vec{0}+\vec{x} = \vec{x}$ in $\mathbb{R}^n$.

(5) $-\vec{x} \in V$ because $A(-\vec{x}) = -\vec{0} = \vec{0}$.

and $-\vec{x}+\vec{x} = \vec{0}$.

(6) $A\vec{x} = \vec{0} \Rightarrow A(r\vec{x}) = r \cdot A\vec{x} = \vec{0}$

, so $r\vec{x} \in V$.

(7) $(r+s)\cdot\vec{x} = r\vec{x} + s\vec{x}$ in $\mathbb{R}^n$.

(8) $r\cdot(\vec{x}+\vec{y}) = r\vec{x} + r\vec{y}$ in $\mathbb{R}^n$

(9) $(r \cdot s)\cdot\vec{x} = r\cdot(s\cdot\vec{x})$ in $\mathbb{R}^n$.

(10) $1\cdot\vec{x} = \vec{x}$ in $\mathbb{R}^n$.

② If $\vec{b} \neq \vec{0}$, then $\vec{b}+\vec{b} \neq \vec{b}$.

Suppose $\vec{x}, \vec{y} \in V$. Then $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$.

However

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{b}+\vec{b} \neq \vec{b}$.

$\Rightarrow \vec{x}+\vec{y} \notin V$. and thus (1) fails.

5. Verify all 10 properties:

Suppose $X_1, X_2, X_3 \in V$ and $r, s \in \mathbb{R}$.

(1) $AX_1 = 0$, $AX_2 = 0$

$\Rightarrow A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$.

so $X_1 + X_2 \in V$.

(2) $X_1 + X_2 = X_2 + X_1$ in $M_{n\times n}$

(3) $(X_1 + X_2) + X_3 = X_1 + (X_2 + X_3)$ in $M_{n\times n}$

(4) $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n\times n}$

and $A\,0 = 0$, so $0 \in V$.

(5) $-X_1 \in V$ because $A(-X_1) = -AX_1 = -0 = 0$.

(6) $AX_1 = 0 \Rightarrow A(rX_1) = rAX_1 = 0$.

so $rX_1 \in V$.

(7) $(r+s)\cdot X_1 = rX_1 + sX_1$ in $M_{n\times n}$

(8) $r(X_1 + X_2) = rX_1 + rX_2$ in $M_{n\times n}$

(9) $(r\cdot s)\cdot X_1 = r\cdot(s\cdot X_1)$ in $M_{n\times n}$

(10) $1\cdot X_1 = X_1$ in $M_{n\times n}$.

6\. (a). ~~$\begin{pmatrix}1\\1\\1\end{pmatrix}$ in~~ +

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are in the space because $1+0+0=1$, $0+1+0=1$

But $\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$ is not in the space because $1+1+0=2 \neq 1$.

$\Rightarrow$ (1) fails.

(b) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ are in the space because $1^2+0^2+0^2=1$, $0^2+1^2+0^2=1$.

But $\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$ is not in the space because $1^2+1^2+0^2=2 \neq 1$.

$\Rightarrow$ (1) fails.

(c). Pick $\underset{\text{vector}}{5} \in \mathbb{R}^+$ and a scalar $-1 \in \mathbb{R}$.

Then $(-1) \cdot 5 = -5 \notin \mathbb{R}^+$.

$\Rightarrow$ (6) fails.

7. Claim: $(\mathbb{R}^+, \oplus, \otimes)$ is a vector space

when $x \oplus y = x \cdot y$ and $r \otimes x = x^r$.

Verify all 10 properties:

Suppose $x, y, z \in \mathbb{R}^+$, $r, s \in \mathbb{R}$.

(1) $x, y > 0$

then $x \oplus y = x \cdot y > 0$

$\Rightarrow x \oplus y \in \mathbb{R}^+$.

(2) $x \oplus y = x \cdot y = y \cdot x = y \oplus x$.

(3) $(x \oplus y) \oplus z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \oplus (y \oplus z)$.

(4) 1 has the property that

$1 \oplus x = 1 \cdot x = x$

$\Rightarrow$ 1 is the zero vector in $\mathbb{R}^+$.

(5). If $x \in \mathbb{R}^+$, then $\frac{1}{x} \in \mathbb{R}^+$.

Also $\frac{1}{x} \oplus x = \frac{1}{x} \cdot x = 1 \leftarrow$ zero in $\mathbb{R}^+$.

so $\frac{1}{x}$ is the additive inverse of x under $\oplus$.

(6) $x > 0 \Rightarrow x^r > 0$

then $r \otimes x = x^r \in \mathbb{R}^+$.

(7) $(r+s) \otimes x = x^{r+s} = x^r \cdot x^s = (r \otimes x) \oplus (s \otimes x)$

addition in $\mathbb{R}$ addition in $\mathbb{R}^+$

(8) $r \otimes (x \oplus y) = (x \cdot y)^r = x^r \cdot y^r = \cancel{r \otimes x}$ $(r \otimes x) \oplus (r \otimes y)$.

(9) $(r \cdot s) \otimes x = x^{rs} = (x^s)^r = r \otimes (s \otimes x)$

multiplication in $\mathbb{R}$ scalar multiplication

(10) The scalar $1 \in \mathbb{R}$ has the property

$1 \otimes x = x^1 = x$.