Sample Solutions for Sample Questions 6.

1. (a). $\begin{pmatrix} 1 & 2 & 4 \\ -3 & 2 & -4 \\ 5 & 4 & 14 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 8\cancel{6} & 8 \\ 0 & -6 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 8\cancel{6} & 8 \\ 0 & 0 & 0\cancel{2} \end{pmatrix}$

has ~~no~~ free variable ~~$\Rightarrow$ unique solution for~~

$$\cancel{\begin{pmatrix} 1 & 2 & 4 \\ -3 & 2 & -4 \\ 5 & 4 & 14 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$\Rightarrow$ S is Not linearly independent.

(Nov 11 更正)

(b). $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 7 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & 0 & 0 \end{pmatrix}$

2×3 matrix must have free variable

$\Rightarrow$ not linearly independent.

(c). $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

no free variable $\Rightarrow$ S is linearly independent.

2.

(a) S is linearly indep.

Suppose $a \cdot \cos(x) + b \cdot \sin(x) = 0$.

Let $x=0 \Rightarrow a \cdot 1 + b \cdot 0 = 0 \Rightarrow a = 0$.

Let $x = \frac{\pi}{2} \Rightarrow a \cdot 0 + b \cdot 1 = 0 \Rightarrow b = 0$.

(b) S is linearly indep.

Suppose $a \cdot 1 + b \cdot \sin(x) + c \cdot \sin(2x) = 0$.

Let $x=0 \Rightarrow a = 0$.

Let $x = \frac{\pi}{2} \Rightarrow b \cdot 1 + c \cdot 0 = 0 \Rightarrow b = 0$.

Let $x = \frac{\pi}{4} \Rightarrow c = 0$.

(c) No, S is not linearly indep.

Because $1 = \cos^2 x + \sin^2 x$.

Equivalently, $1 \cdot 1 - 1 \cdot \cos^2 x - 1 \cdot \sin^2 x = 0$.

(d) No, S is not linearly indep.

Because $\cos(2x) = \cos^2 x - \sin^2 x$.

Equivalently, $-1 \cdot \cos(2x) + 1 \cdot \cos^2 x - 1 \cdot \sin^2 x = 0$.

3.

Let $\{\vec{v}_1, \ldots, \vec{v}_{n+1}\}$ be vectors in $\mathbb{R}^n$.

Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{n+1} \\ | & & | \end{pmatrix}$ be an $n \times (n+1)$ matrix.

The reduced echelon form of A

has at most n leading variables.

But there are $n+1$ columns

$\Rightarrow$ there must be some free variable(s).

So $\{\vec{v}_1, \ldots, \vec{v}_{n+1}\}$ is linearly dependent.

4. Claim: $S \cup \{\vec{v}\}$ is linearly indep $\Leftrightarrow$ $\vec{v} \notin \mathrm{span}(S)$.

"$\Rightarrow$" Let $T = S \cup \{\vec{v}\}$.

By definition, T is linearly indep. means

$\vec{v} \notin \mathrm{span}(T \setminus \{\vec{v}\}) = \mathrm{span}(S)$.

"$\Leftarrow$" Suppose $c_0 \cdot \vec{v} + c_1 \vec{s}_1 + \ldots + c_k \cdot \vec{s}_k = \vec{0}$.

If $c_0 \neq 0$, then

$\vec{v} = -\frac{1}{c_0}(c_1 \vec{s}_1 + \ldots + c_k \vec{s}_k) \in \mathrm{span}(S)$,

a contradiction.

$\Rightarrow$ $c_0 = 0$ and $c_1 \cdot \vec{s}_1 + \ldots + c_k \cdot \vec{s}_k = \vec{0}$

Since S is linearly indep., $c_1 = c_2 = \ldots = c_k = 0$.

We've showed that $c_0 = c_1 = \ldots = c_k = 0$

$\Rightarrow$ $S \cup \{\vec{v}\}$ is linearly indep.

5.

① Claim: If S is linearly dep. and $\hat{S} \supseteq S$, then $\hat{S}$ is linearly dep.

S is linearly dep.

$\Rightarrow c_1\vec{s}_1 + \dots + c_k\vec{s}_k = 0$ for some $c_1, \dots, c_k \in \mathbb{R}$ not all zero, $\vec{s}_1, \dots, \vec{s}_k \in S$

Since $\hat{S} \supseteq S$, $\vec{s}_1, \dots, \vec{s}_k \in \hat{S}$.

Now $c_1\vec{s}_1 + \dots + c_k \cdot \vec{s}_k = 0$ with $c_1, \dots, c_k$ not all zero, $\vec{s}_1, \dots, \vec{s}_k \in \hat{S}$.

$\Rightarrow \hat{S}$ is linearly dep.

② Claim: If S is linearly indep. and $\hat{S} \subseteq S$, then $\hat{S}$ is linearly indep.

Suppose $c_1 \cdot \vec{s}_1 + \dots + c_k\vec{s}_k = \vec{0}$ with $c_1, \dots, c_k \in \mathbb{R}$, $\vec{s}_1, \dots, \vec{s}_k \in \hat{S}$.

Since $\hat{S} \subseteq S$, $\vec{s}_1, \dots, \vec{s}_k \in S$.

But S is linearly indep. $\Rightarrow c_1 = \dots = c_k = 0$.

$\Rightarrow \hat{S}$ is linearly indep.

6. Suppose $c_1\vec{v_1} + \dots + c_n\vec{v_n} = \vec{0}$ with $c_1, \dots, c_n \in \mathbb{R}$.

Then $(c_1\vec{v_1} + \dots + c_n\vec{v_n}) \cdot \vec{v_i} = \vec{0} \cdot \vec{v_i} = \cancel{0}\, 0$.

$$\Rightarrow \quad c_i \cdot |\vec{v_i}|^2 = 0.$$

Since $\vec{v_i}$ is nonzero $\Rightarrow$ $|\vec{v_i}|^2 \neq 0$

$$\Rightarrow \quad c_i = 0.$$

Do this for each $i = 1, \dots, n$.

$\Rightarrow$ $c_1 = \dots = c_n = 0$

So $\{\vec{v_1}, \dots, \vec{v_n}\}$ is linearly indep.

7. Let $A = \begin{pmatrix} 1^* & & ? \\ & \ddots & \\ O & & 1^* \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_n} \\ | & & | \end{pmatrix}$.

Suppose $c_1 \cdot \vec{v_1} + \dots + c_n\vec{v_n} = \vec{0}$.

Look at the n-th entry:

$$c_1 \cdot 0 + \dots + c_{n-1} \cdot 0 + c_n \cdot a_{n,n} = 0.$$

Since $a_{n,n} \neq 0 \Rightarrow c_n = 0$.

Now $c_1 \cdot \vec{v_1} + \dots + c_{n-1} \cdot \vec{v}_{n-1} = \vec{0}$.

Look at the $(n-1)$-th entry:

$$c_1 \cdot 0 + \dots + c_{n-2} \cdot 0 + c_{n-1} \cdot a_{n-1,n-1} = 0.$$

Again, $a_{n-1,n-1} \neq 0 \Rightarrow c_{n-1} = 0$.

Keep doing this for $n-2, n-3, \dots, 1$ $\Rightarrow$ $c_n = c_{n-1} = \dots = c_1 = 0$

$\Rightarrow$ Columns of A form a linearly indep. set.

The case for rows is similar. Try it.

You will find $c_1 = 0$, $c_2 = 0$, ..., and then $c_n = 0$.