

Sample Solutions for Sample Questions 7.

$$4 \quad (a). \quad \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$
$$E_1, E_2, E_3, E_4.$$

Note that

$$\text{all matrices} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$

$$= \left\{ aE_1 + bE_2 + cE_3 + dE_4 \mid a, b, c, d \in \mathbb{R} \right\}.$$

$$= \text{span}(\{E_1, E_2, E_3, E_4\}).$$

And they are indep.

$$(b) \quad \langle 1, x, x^2, x^3 \rangle$$

$$\text{all polynomials of deg} \leq 3 = \{a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R}\}$$

$$= \text{span}(\{1, x, x^2, x^3\}).$$

And they are indep.

(c). Solve $c - 2b = 0$ first. Note a is also a free variable.

Let $b = s$.
 $a = t$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} t \\ s \\ 2s \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

So the set = $\left\{ t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + s \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$.

$$= \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}$$

They are indep.

So $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\rangle$ is a basis.

(d)

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

\Rightarrow so $\left\langle \vec{v}_1, \vec{v}_3 \right\rangle$ is a basis

[make sure you understand why?]

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

(e)

$$\begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 1 & 1 & 8 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \left\langle \vec{v}_1, \vec{v}_2, \vec{v}_4 \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \right\rangle$ is a basis.

(f)

$$\begin{pmatrix} 1 & -4 & 3 & -1 \\ 2 & -8 & 6 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↑ ↑ ↑
free variables.

Solve $\beta_1, \beta_2, \beta_3$.

$$\beta_1: \text{Let } x_2=1, x_3=0, x_4=0 \Rightarrow \begin{pmatrix} +4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\beta_2: \text{Let } x_2=0, x_3=1, x_4=0 \Rightarrow \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_3: \text{Let } x_2=0, x_3=0, x_4=1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} +4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the solution set.

(Nov 11 更正)

注意:

(a), (b) (c), (f) (d), (e) 各為不同類型.

(a), (b): 找常見空間的基底。各個空間有個標準答案
eg. \mathbb{R}^n 有標準基底.

(c), (f): 找解空間的基底。用 $\beta_1, \beta_2, \dots, \beta_k$, 其中 k 是
自由變數的個數.

(d), (e): 找一個 span 的基底。用對應到領導變數的
那些 column.

2.

① Claim: $\langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$.

~~Since $c_1 \vec{x}_1 \in \text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$~~

~~$c_2 \vec{x}_2$~~
 ~~$c_3 \vec{x}_3$~~

Note $\vec{x}_1 = \frac{1}{c_1} \cdot (c_1 \vec{x}_1) \in \text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$.

Similarly $\vec{x}_2, \vec{x}_3 \in \text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$.

So $\text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle \subseteq \text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$.

Also, ~~$c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3$~~ $c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \in \text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$

so $\text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle \subseteq \text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$.

In conclusion, $\text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle = \text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$.

Now $\text{span} \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle = \text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$

↑
3-dimensional space.

$\Rightarrow \langle c_1 \vec{x}_1, c_2 \vec{x}_2, c_3 \vec{x}_3 \rangle$ is a basis.

also three vectors

② Claim: $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ is a basis.

Similarly, $\vec{y}_1 = \vec{x}_1 + \vec{x}_1$

$\vec{y}_2 = \vec{x}_1 + \vec{x}_2$

$\vec{y}_3 = \vec{x}_1 + \vec{x}_3$

all in $\text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$

$\vec{x}_1 = \frac{1}{2} \vec{y}_1$

$\vec{x}_2 = \vec{y}_2 - \frac{1}{2} \vec{y}_1$

$\vec{x}_3 = \vec{y}_3 - \frac{1}{2} \vec{y}_1$

all in $\text{span} \langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$

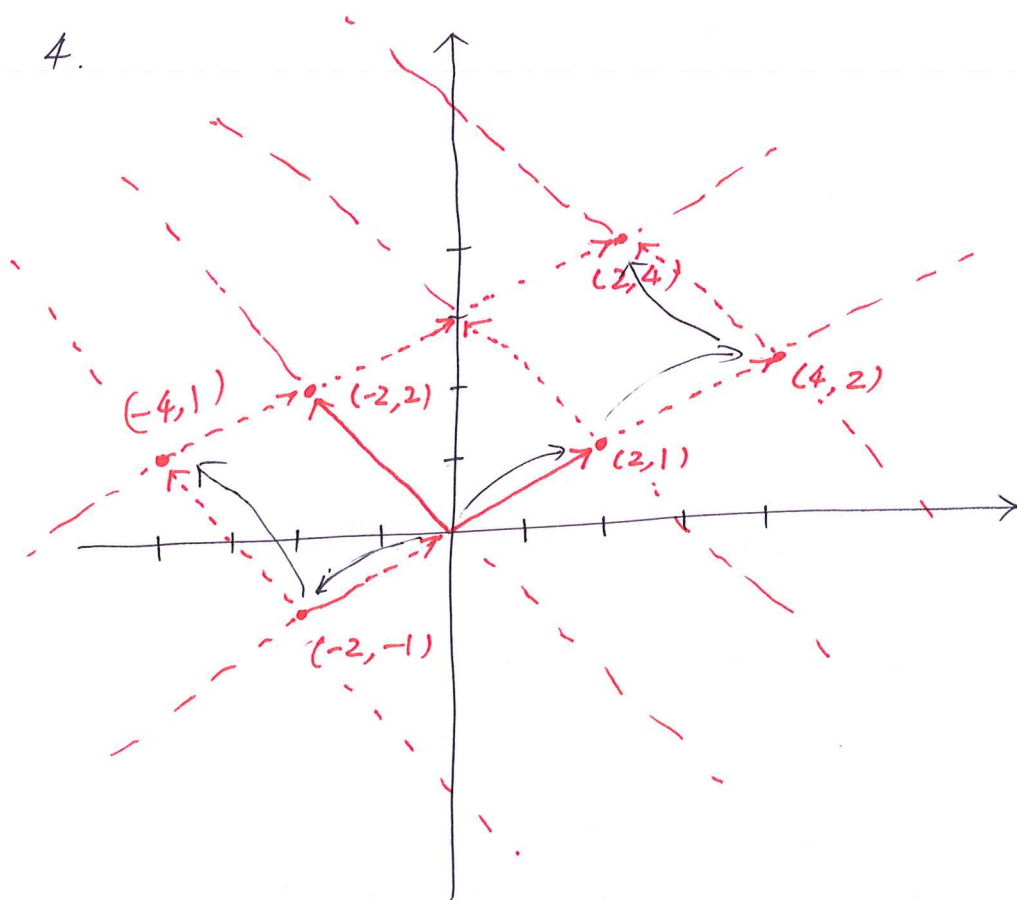
So $\text{span} \langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle = \text{span} \langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$.

Since $\langle \vec{x}_1, \vec{x}_2, \vec{x}_3 \rangle$ is a basis and the spanning space has dimension 3, and $\langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ has 3 vectors spanning the same space $\Rightarrow \langle \vec{y}_1, \vec{y}_2, \vec{y}_3 \rangle$ is a basis.

3. all symmetric 3×3 matrices = $\left\{ \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{R} \right\}$.

\Rightarrow basis = $\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$

4.



So $\text{Rep}_B \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

meaning $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$\text{Rep}_B \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

meaning $\begin{pmatrix} -4 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

5.

[Key ; find \vec{v} not in the spanning set.]

(a). Find $\vec{v} \notin \text{span}(\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \})$.

e.g. \vec{v} can be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b). Find $\vec{v} \notin \text{span}(\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \})$.

e.g. \vec{v} can be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c). Find $\vec{v} \in \text{span}(\{ x, 1+x^2 \})$.

e.g. \vec{v} can be x^2 .

6. Suppose $x_{f_1}, x_{f_2}, \dots, x_{f_k}$ be the k free variables.

So we set $x_{f_i} = 1$ and $x_{f_j} = 0$ for all $j \neq i$ to find $\vec{\beta}_i$.

This means the f_i -entry of $\vec{\beta}_j$ is 0 when $j \neq i$
and is 1 when $j = i$.

~~Let $(\beta_j)_i$ be the i -th entry of~~

Use the notation $(\vec{v})_i$ to denote the i -th entry of \vec{v} .

Suppose $c_1 \vec{\beta}_1 + \dots + c_k \vec{\beta}_k = 0$.

$$\Rightarrow (c_1 \vec{\beta}_1)_{f_1} + \dots + (c_k \vec{\beta}_k)_{f_1} = 0 \Rightarrow c_1 = 0.$$

$\begin{matrix} \parallel & & \parallel & & \parallel \\ c_1 & & 0 & & 0 \end{matrix}$

Similarly, $(c_1 \vec{\beta}_1)_{f_i} + \dots + (c_i \vec{\beta}_i)_{f_i} + \dots + (c_k \vec{\beta}_k)_{f_i} = 0 \Rightarrow c_i = 0$.

$\begin{matrix} \parallel & & \dots & & \parallel & & \parallel & & \dots & & \parallel \\ 0 & & \dots & & 0 & & c_i & & 0 & & \dots & & 0 \end{matrix}$

$\Rightarrow c_1 = \dots = c_k = 0 \Rightarrow S$ is linearly indep.

