Sample Questions 8

1. Let
$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{bmatrix}.$$
For each of the row space, the column space, and the null space of $\mathbf{A}$, find a basis and determine the dimension.

2. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{bmatrix}.$$
For each of the row space, the column space, and the null space of $\mathbf{A}$, find a basis and determine the dimension.

3. Given $a, b, c \in \mathbb{R}$ with $a \neq 0$, find a value of d so that the rank of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 1.

4. A linear system is said to be *consistent* if there is at least one solution. Prove that a linear system $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if that the rank of $\mathbf{A}$ is the same as the rank of the augmented matrix $\left[\ \mathbf{A} \mid \mathbf{b}\ \right]$.

5. An $m \times n$ matrix has *full row rank* if its row rank is m, and it has *full column rank* if its column rank is n. Let $\mathbf{A}$ be an $m \times n$ matrix.
   (a) Prove that $\mathbf{Ax} = \mathbf{b}$ is consistent for any $\mathbf{b} \in \mathbb{R}^m$ if and only if $\mathbf{A}$ has full row rank.
   (b) Prove that $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^m$ making $\mathbf{Ax} = \mathbf{b}$ consistent if and only if $\mathbf{A}$ has full column rank.

6. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices. Prove that
$$\operatorname{rank}(\mathbf{A} + \mathbf{B}) \leqslant \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B})$$
and
$$\operatorname{rank}(\mathbf{AB}) \leqslant \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}.$$

7. Use the Exchange Lemma to prove the "Extension Lemma": Let B be a basis of a space V with $|B| = n$. If D is a linearly independent set with $|D| = m$ and $m < n$, then there is a subset $B' \subseteq B$ with $|B'| = n - m$ such that $D \cup B'$ is a basis. (This is another way to show every basis has the same size, so do not use Theorem Two.III.2.5.)