

Sample Solution for Sample Questions 9.

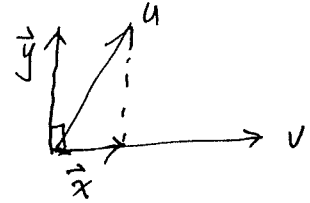
1. 公式: \vec{u} 在 \vec{v} 上的投影為 $\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 1 + (-1) + 1 = 1$$

$$|\vec{v}|^2 = 1^2 + 1^2 + 1^2 = 3.$$

$$\Rightarrow \vec{x} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \frac{1}{3} \vec{v} \quad (\text{投影})$$

$$\vec{y} = \vec{u} - \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix}$$



[You may check that $\langle \vec{v}, \vec{y} \rangle = 0$.]

2. Suppose $A = [a_{ij}]$, $B = [b_{ij}]$.

Then

$$i, j\text{-entry of } (AB)^T = j, i\text{-entry of } AB$$

$$= \sum_{k=1}^n a_{jk} b_{ki}$$

$$i, j\text{-entry of } B^T A^T = \sum_{k=1}^n (B^T)_{ik} (A^T)_{kj}$$

$$= \sum_{k=1}^n b_{ki} a_{kj}$$

the same.

3.

$$Ax = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \langle Ax, y \rangle = [3 \ 2 \ 1] \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -12$$

$$A^T y = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 20 \end{pmatrix}$$

$$\Rightarrow \langle x, A^T y \rangle = [14 \ 20] \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -12$$

Thus $\langle Ax, y \rangle = -12 = \langle x, A^T y \rangle$.

4. Claim: $A\vec{x} = \vec{0} \iff A^T A\vec{x} = \vec{0}$.

" \implies " Suppose $A\vec{x} = \vec{0}$

$$\text{Then } A^T A\vec{x} = A^T(A\vec{x})$$

$$= A^T \vec{0} = \vec{0}$$

$\uparrow \quad \uparrow$
長度不同的零向量

" \impliedby " Suppose $A^T A\vec{x} = \vec{0}$.

$$\text{Then } \langle \vec{x}, A^T A\vec{x} \rangle = \langle \vec{x}, \vec{0} \rangle = 0$$

$$\Rightarrow \langle A\vec{x}, A\vec{x} \rangle = 0$$

$$\Rightarrow |A\vec{x}|^2 = 0$$

$$\Rightarrow A\vec{x} = \vec{0}$$

注意: A^T 不一定是方阵
也不一定有反矩阵, 所以
不用說 ~~A^{-1}~~

$$A^T A\vec{x} = A^T \vec{0}$$

$$\downarrow$$

$$A\vec{x} = \vec{0}$$

5. Let A be an $m \times n$ matrix. Then $A^T A$ is $n \times n$.

Claim: A has full column rank $\Leftrightarrow A^T A$ is invertible.

Recall: $\textcircled{1}$ A has full column rank

\Leftrightarrow no free variable

$\Leftrightarrow \vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

$\textcircled{2}$ $A^T A$ is invertible

$\Leftrightarrow A^T A$ is nonsingular

$\Leftrightarrow \vec{x} = \vec{0}$ is the only solution of $A^T A \vec{x} = \vec{0}$.

Claim:

$\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$

$\Leftrightarrow \vec{x} = \vec{0}$ is the only solution of $A^T A \vec{x} = \vec{0}$.

pf. \Rightarrow

Suppose $A^T A \vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \vec{0}$ [By Problem 4]

$\Rightarrow \vec{x} = \vec{0}$ [By assumption].

\Leftarrow

Suppose $A\vec{x} = \vec{0}$

$\Rightarrow A^T A \vec{x} = \vec{0}$ [By Problem 4]

$\Rightarrow \vec{x} = \vec{0}$ [By assumption].

6.

$$\langle x, y \rangle = \overline{(3+4i)} \cdot (3+2i) + \overline{(-4i)} (2-3i) + \overline{(2-i)} \cdot i$$

$$= (3-4i)(3+2i) + (4i)(2-3i) + (2+i) \cdot i$$

$$= 17 - 6i + \overset{12}{\cancel{12}} + \overset{8i}{\cancel{8i}} - 1 + 2i$$

$$= \cancel{22} 28 + 4i$$

$$\langle y, x \rangle = \overline{(3+2i)} (3+4i) + \overline{(2-3i)} (-4i) + \overline{i} \cdot (2-i)$$

$$= (3-2i)(3+4i) + (2+3i)(-4i) + (-i)(2-i)$$

$$= 17 + 6i + 12 + 8i - 1 - 2i$$

$$= 28 - 4i \quad \leftarrow \overline{\langle x, y \rangle}$$

$$|x| = \sqrt{\langle x, x \rangle}$$

$$= \sqrt{\overline{(3+2i)}(3+2i) + \overline{(2-3i)}(2-3i) + \overline{i}i}$$

$$= \sqrt{3^2 + 2^2 + 2^2 + 3^2 + 1^2}$$

$$= \sqrt{27}$$

7. Since $f(x)$ passes through the 5 points,

$$f(p_1) = a + bp_1 + cp_1^2 + dp_1^3 + ep_1^4 = q_1,$$

\vdots

$$f(p_5) = a + bp_5 + cp_5^2 + dp_5^3 + ep_5^4 = q_5$$



$$\begin{pmatrix} 1 & p_1 & p_1^2 & p_1^3 & p_1^4 \\ 1 & p_2 & p_2^2 & p_2^3 & p_2^4 \\ 1 & p_3 & p_3^2 & p_3^3 & p_3^4 \\ 1 & p_4 & p_4^2 & p_4^3 & p_4^4 \\ 1 & p_5 & p_5^2 & p_5^3 & p_5^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$$

