Math555 Homework 3

Note: You may turn in your homework through paper work (first three weeks only) or through CoCalc. To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Suppose $\pi = b_1 b_2 \cdots b_n$. Recall that the inversion table of π is $a_1 a_2 \cdots a_n$ such that $0 \leqslant a_i \leqslant n - i$ for all i, where

$$a_{b_i} = \left|\{j < i : b_j > b_i\}\right|.$$

A left-to-right maximum of π is a digit b_j such that $b_j \geqslant b_i$ for all $i \leqslant j$. Finish the following table.

Solution. The table below lists all the 24 permutations in Σ_4 and their inversion tables.

permutations in Σ_4	inversion table	# of left-to-right maxima
1234	0000	4
1243	0010	3
1324	0100	3
1342	0200	3
1423	0110	2
1432	0210	2
2134	1000	3
2143	1010	2
2314	2000	3
2341	3000	3
2413	2010	2
2431	3010	2
3124	1100	2
3142	1200	2
3214	2100	2
3241	3100	2
3412	2200	2
3421	3200	2
4123	1110	1
4132	1210	1
4213	2110	1
4231	3110	1
4312	2210	1
4321	3210	1

2. Given that $s(n,k) = (-1)^{n-k}c(n,k)$ and

$$\sum_{k=0}^{n} c(n,k)x^k = (x+n+1)_n,$$

show that

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n.$$

Solution. This follows from direct computation.

$$\begin{aligned}
\sum_{k=0}^{n} s(n,k)x^k &= \sum_{k=0}^{n} (-1)^{n-k}c(n,k)x^k \\
&= (-1)^n \sum_{k=0}^{n} c(n,k)(-x)^k \\
&= (-1)^n(-x+n-1)_n \\
&= (-1)^n(-x+n-1)(-x+n)\cdots(-x) \\
&= (x)(x-1)\cdots(x-n+1) = (x)_n.
\end{aligned}$$

In other words, $s(n,k)$ is the coefficient of x^k in $(x)_n$,