

Math555 Homework 5

Note: To submit the k -th homework, simply put your files in the folder HW k on CoCalc, and it will be collected on the due day.

1. Recall that \mathcal{C}_n is the $n \times n$ board where rooks are only allowed on the positions

$$\{(i, i), (i, i + 1) : i = 1, \dots, n\}.$$

Consider the cycle graph C_{2n} where vertices are $X \cup Y$ with

$$X = \{x_i : i = 1, \dots, n\} \text{ and } Y = \{y_i : i = 1, \dots, n\}$$

and edges are

$$E = \{(x_i, y_i), (x_i, y_{i+1}) : i = 1, \dots, n\}.$$

All subscripts are taking modulo n . A k -matching on a graph means a set of k edges such that none of them share a same vertex. Show that the number of k -matchings on C_{2n} is the number of ways to put k rooks on \mathcal{C}_n in non-attacking positions.

2. Let $\phi(n)$ be the Euler's totient function. That is, $\phi(n)$ is the number of integers k with $\gcd(k, n) = 1$ and $1 \leq k \leq n$. Consider $n = 12$ and the set $[n] = \{1, \dots, 12\}$. Let

$$A_d = \{k \in [n] : \gcd(k, n) = d\}.$$

For each $d \mid n$, find A_d and verify $|A_d| = \phi(n/d)$.