Math555 Homework 5

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Recall that $\mathcal{C}_n$ is the $n \times n$ board where rooks are only allowed on the positions

$$\{(i,i),(i,i+1) : i = 1,\dots,n\}.$$

Consider the cycle graph C_{2n} where vertices are $X \cup Y$ with

$$X = \{x_i : i = 1,\dots,n\} \text{ and } Y = \{y_i : i = 1,\dots,n\}$$

and edges are

$$E = \{(x_i,y_i),(x_i,y_{i+1}) : i = 1,\dots,n\}.$$

All subscripts are taking modulo n. A k-matching on a graph means a set of k edges such that none of them share a same vertex. Show that the number of k-matchings on C_{2n} is the number of ways to put k rooks on $\mathcal{C}_n$ in non-attacking positions.

Solution. Let M be a k-matching on C_{2n}. Then

$$\{(i,j) : (x_i,y_j) \in M\}$$

is a non-attacking rook placement on $\mathcal{C}_n$. Conversely, if $\mathcal{P}$ is a non-attacking placement on $\mathcal{C}_n$, then

$$\{(x_i,y_j) : (i,j) \in \mathcal{P}\}$$

forms a k-matching of C_{2n}.

2. Let $\phi(n)$ be the Euler's totient function. That is, $\phi(n)$ is the number of integers k with $\gcd(k,n) = 1$ and $1 \leqslant k \leqslant n$. Consider $n = 12$ and the set $[n] = \{1,\ldots,12\}$. Let

$$A_d = \{k \in [n] : \gcd(k,n) = d\}.$$

For each $d \mid n$, find A_d and verify $|A_d| = \phi(n/d)$.

Solution. The value of d can be $1,2,3,4,6,12$.

$$\begin{aligned}
A_1 &= \{1,5,7,11\} \\
A_2 &= \{2,10\} \\
A_3 &= \{3,9\} \\
A_4 &= \{4,8\} \\
A_6 &= \{6\} \\
A_{12} &= \{12\}
\end{aligned}$$

They verify the $|A_d| = \phi(n/d)$.

$$\begin{aligned}
\phi(12/1) &= \phi(12) = \phi(3)\cdot\phi(4) = 2\cdot 2 = 4 \\
\phi(12/2) &= \phi(6) = \phi(2)\cdot\phi(3) = 1\cdot 2 = 2 \\
\phi(12/3) &= \phi(4) = 2 \\
\phi(12/4) &= \phi(3) = 2 \\
\phi(12/6) &= \phi(2) = 1 \\
\phi(12/12) &= \phi(1) = 1
\end{aligned}$$