Math555 Midterm 1

Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Use Newton's binomial theorem to prove that

$$(1-4x)^{-1/2} = \sum_{n \geqslant 0} \binom{2n}{n} x^n.$$

Then use it to prove that, for every positive integer n,

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n.$$

2. [4pt] Prove that if $j \geqslant 2$, then

$$\ln j \geqslant \int_{j-1/2}^{j+1/2} \ln x \, dx,$$

and use it to prove that

$$n! \geqslant \sqrt{\frac{8e^2 n}{27}} \left(\frac{n}{e}\right)^n.$$

3. [4pt] Let $p_k(n)$ denote the number of partitions of integer n into k parts. Fix an integer $t \geqslant 0$. Show that as $n \to \infty$, $p_{n-t}(n)$ becomes eventually constant value $f(t)$. What is the $f(t)$? What is the least value of n for which $p_{n-t}(n) = f(t)$?

   [Recall that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.]

4. [4pt] Let
$$x^{(n)} = (x+n-1)_n = x(x+1)\cdots(x+n-1)$$
be the n-th *rising factorial* of x. Recall that
$$\sum_{k\geqslant 0} c(n,k)x^k = x^{(n)},$$
so $c(n,k)$ are the coefficoents that change the basis from $\mathcal{B}_1 = \{x^{(0)}, x^{(1)}, \ldots\}$ to $\mathcal{B}_2 = \{x^0, x^1, \ldots, \}$. Find the coefficients that change the basis from $\mathcal{B}_2$ to $\mathcal{B}_1$. In other words, find the coefficients $A_{n,k}$ such that
$$\sum_{k\geqslant 0} A_{n,k}x^{(k)} = x^n.$$
[Hint: You may use the Stirling numbers.]

5. [4pt] Let N and X be two sets with $|N| = |X| = 3$. Fill in the following table by the number of functions $f : N \to X$ with the given conditions.

N	X	Any f	injective f	surjective f
dist	dist	(i)	(ii)	(iii)
indist	dist	(iv)	(v)	(vi)
dist	indist	(vii)	(viii)	(ix)
indist	indist	(x)	(xi)	(xii)