Math555 Midterm 1

Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Use Newton's binomial theorem to prove that

$$(1-4x)^{-1/2} = \sum_{n\geqslant 0} \binom{2n}{n} x^n.$$

Then use it to prove that, for every positive integer n,

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n.$$

Solution. By Newton's binomial theorem

$$(1-4x)^{-1/2} = \sum_{n\geqslant 0} \binom{-1/2}{n} (-4)^n x^n.$$

We may compute

$$\begin{aligned}\binom{-1/2}{n}(-4)^n &= (-1)^n 2^{2n} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} \\ &= 2^n \cdot \frac{(1)(3)\cdots(2n-1)}{n!} \\ &= \frac{(2)(4)\cdots(2n)}{n!} \cdot \frac{(1)(3)\cdots(2n-1)}{n!} \\ &= \binom{2n}{n}.\end{aligned}$$

Consider

$$\begin{aligned}(1-4x)^{-1} &= \left(\sum_{n\geqslant 0}\binom{2n}{n}x^n\right)\left(\sum_{n\geqslant 0}\binom{2n}{n}x^n\right) \\ &= \left(\sum_{i\geqslant 0}\binom{2i}{i}x^n\right)\left(\sum_{i\geqslant 0}\binom{2j}{j}x^n\right).\end{aligned}$$

To see the desired equality, compare the coefficients of x^n on both sides.

2. [4pt] Prove that if $j \geqslant 2$, then

$$\ln j \geqslant \int_{j-1/2}^{j+1/2} \ln x \, dx,$$

and use it to prove that

$$n! \geqslant \sqrt{\frac{8e^2 n}{27}} \left(\frac{n}{e}\right)^n.$$

Solution. Since $\ln x$ is concave down, it is below its tangent lines. The tangent line of $\ln x$ at j is $y = \frac{1}{j}(x-j) + \ln(j)$, so

$$\ln j = \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{1}{j}(x-j) + \ln j \; dx \geqslant \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \ln x \; dx$$

for $j > \frac{1}{2}$. Therefore,

$$\begin{aligned} \ln(n!) = \sum_{j=2}^{n} \ln j &\geqslant \int_{\frac{3}{2}}^{n+\frac{1}{2}} \ln x \; dx \\ &= (n+0.5)\ln(n+0.5) - 1.5\ln 1.5 - (n-1). \end{aligned}$$

So

$$n! \geqslant \frac{(n+0.5)^{n+0.5}}{1.5^{1.5} e^{n-1}} \geqslant \sqrt{\frac{8e^2 n}{27}} \left(\frac{n}{e}\right)^n.$$

3. [4pt] Let $p_k(n)$ denote the number of partitions of integer n into k parts. Fix an integer $t \geqslant 0$. Show that as $n \to \infty$, $p_{n-t}(n)$ becomes eventually constant value $f(t)$. What is the $f(t)$? What is the least value of n for which $p_{n-t}(n) = f(t)$?

[Recall that $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.]

Solution. By the recurrence relation,

$$p_{n-t}(n) = p_{n-1-t}(n-1) + p_{n-t}(t).$$

This means $p_{n-t}(n)$ is an increasing sequence in terms of n. Observe that

$$p_{n-t}(t) \begin{cases} \geqslant 0 & \text{if } n-t \leqslant t, \\ = 0 & \text{if } n-t > t. \end{cases}$$

Therefore, for fixed t, the sequence $p_{n-t}(n)$ is strictly increasing until $n = 2t$ and $p_{n-t}(n) = f(t)$ for all $n \geqslant 2t$. Here

$$f(t) = p_t(2t) = \sum_{k=1}^{t} p_k(t).$$

4. [4pt] Let

$$x^{(n)} = (x+n-1)_n = x(x+1)\cdots(x+n-1)$$

be the n-th *rising factorial* of x. Recall that

$$\sum_{k\geqslant 0} c(n,k)x^k = x^{(n)},$$

so $c(n,k)$ are the coefficoents that change the basis from $\mathcal{B}_1 = \{x^{(0)}, x^{(1)}, \ldots\}$ to $\mathcal{B}_2 = \{x^0, x^1, \ldots,\}$. Find the coefficients that change the basis from $\mathcal{B}_2$ to $\mathcal{B}_1$. In other words, find the coefficients $A_{n,k}$ such that

$$\sum_{k\geqslant 0} A_{n,k}x^{(k)} = x^n.$$

[Hint: You may use the Stirling numbers.]

Solution. Recall that

$$\sum_{k\geqslant 0} S(n,k)(x)_k = x^n.$$

Substitute x by $-x$ and get

$$\sum_{k\geqslant 0} S(n,k)(-x)_k = (-x)^n = (-1)^n x^n.$$

Note that

$$\begin{aligned}(-x)_k &= (-x)(-x-1)\cdots(-x-k+1)\\ &= (-1)^k(x)(x+1)\cdots(x+k-1)\\ &= (-1)^k x^{(k)}.\end{aligned}$$

Therefore,

$$\sum_{k\geqslant 0} (-1)^{n-k} S(n,k)x^{(k)} = x^n,$$

and $A_{n,k} = (-1)^{n-k}S(n,k)$ is the answer.

5. [4pt] Let N and X be two sets with $|N| = |X| = 3$. Fill in the following table by the number of functions $f : N \to X$ with the given conditions.

N	X	Any f	injective f	surjective f
dist	dist	(i)	(ii)	(iii)
indist	dist	(iv)	(v)	(vi)
dist	indist	(vii)	(viii)	(ix)
indist	indist	(x)	(xi)	(xii)

Solution. Use the formulas given in the lecture notes.

N	X	Any f	injective f	surjective f
dist	dist	$3^3 = 27$	$(3)_3 = 6$	$3!S(3,3) = 6$
indist	dist	$\left(\!\binom{3}{3}\!\right) = 10$	$\binom{3}{3} = 1$	$\left(\!\binom{3}{0}\!\right) = 1$
dist	indist	$S(3,0) + S(3,1) + S(3,2) + S(3,3) = 5$	1	$S(3,3) = 1$
indist	indist	$p_0(3) + p_1(3) + p_2(3) + p_3(3) = 3$	1	$p_3(3) = 1$