Math555 Midterm 2

Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Let $\phi(n)$ be the Euler's totient function. That is, $\phi(n)$ is the number of integers k with $\gcd(k, n) = 1$ and $1 \leqslant k \leqslant n$. Consider $n = 16$ and the set $[n] = \{1, \ldots, 16\}$. Let

$$A_d = \{k \in [n] : \gcd(k, n) = d\}.$$

For each $d \mid n$, find A_d and verify $|A_d| = \phi(n/d)$.

Solution. The value of d can be $1, 2, 4, 8, 16$.

$$\begin{aligned} A_1 &= \{1, 3, 5, 7, 9, 11, 13, 15\} \\ A_2 &= \{2, 6, 10, 14\} \\ A_4 &= \{4, 12\} \\ A_8 &= \{8\} \\ A_{16} &= \{16\} \end{aligned}$$

Then verify the $|A_d| = \phi(n/d)$.

$$\begin{aligned} \phi(16/1) &= \phi(16) = 2^4 - 2^3 = 8 \\ \phi(16/2) &= \phi(8) = 2^3 - 2^2 = 4 \\ \phi(16/4) &= \phi(4) = 2 \\ \phi(16/8) &= \phi(2) = 1 \\ \phi(16/16) &= \phi(1) = 1 \end{aligned}$$

2. [4pt] Let $\mu(n)$ be the Möbius function. Prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Solution. Suppose $n = p_1^{a_1} \cdots p_r^{a_r}$ is the prime decomposition of n. Then every $d \mid n$ can be written as $d = p_1^{b_1} \cdots p_r^{b_r}$ with $0 \leqslant b_i \leqslant a_i$. Thus, $\mu(d) = 0$ whenever one of the power b_i is at least 2. Therefore,

$$\begin{aligned} \sum_{d|n} \mu(d) &= \sum_{\substack{0 \leqslant b_i \leqslant 1 \\ \text{for all } i}} \mu(p_1^{b_1} \cdots p_r^{b_r}) \\ &= \sum_{I \subseteq \{1,\dots,r\}} (-1)^{|I|} \\ &= \sum_{k=0,\dots,r} (-1)^k \binom{r}{k} \\ &= (1-1)^r = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \end{aligned}$$

3. [4pt] Suppose x_n is an integer for every $n \geqslant 1$ and

$$y_n = \sum_{d|n} x_d.$$

If $y_n = n^2$ for every $n \geqslant 1$. Use Möbius inversion to find x_{24}.

Solution. By Möbius inversion,

$$\begin{aligned} x_{24} &= y_1\mu(24) + y_2\mu(12) + y_3\mu(8) + y_4\mu(6) \\ &\quad + y_6\mu(4) + y_8\mu(3) + y_{12}\mu(2) + y_{24}\mu(1) \\ &= 0 + 0 + 0 + 16 + 0 - 64 - 144 + 576 = 384. \end{aligned}$$

4. [4pt] Solve the recurrence relation below.

$$\begin{cases} a_n + 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0. \\ a_0 = 4, a_1 = -3, a_2 = 11. \end{cases}$$

Solution. The characteristic polynomial is

$$p(x) = x^3 - 3x - 2 = (x+1)^2(x-2)$$

with the roots $-1, -1, 2$. Thus, the formula for a_n is

$$a_n = A \cdot (-1)^n + B \cdot n(-1)^n + C \cdot 2^n.$$

Substituting this equality with $n = 0, 1, 2$, we get the following equations.

$$\begin{cases} A + C = 4 \\ (-1)A + (-1)B + 2C = -3 \\ A + 2B + 4C = 11 \end{cases}$$

It follows that $A = 3$, $B = 2$, and $C = 1$, so

$$a_n = 3(-1)^n + 2n(-1)^n + 2^n.$$

5. [4pt] Find the closed form of the generating function $\sum_{k\geqslant 0} k^2x^k$.

Solution. It is known that

$$\sum_{k\geqslant 0} x^k = (1-x)^{-1}.$$

Next compute

$$\begin{aligned}\sum_{k\geqslant 0} kx^k &= x\sum_{k\geqslant 1} kx^{k-1}\\ &= x\left(\sum_{k\geqslant 1} x^k\right)'\\ &= x[(1-x)^{-1}-1]' = x(1-x)^{-2}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k\geqslant 0} k^2x^k &= x\sum_{k\geqslant 1} k^2x^{k-1}\\ &= x\left(\sum_{k\geqslant 1} kx^k\right)'\\ &= x[x(1-x)^{-2}]' = x(1-x)^{-2} + 2x^2(1-x)^{-3}.\end{aligned}$$