國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

期末考 January 6, 2020 **Final Examination**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Write down an example of a system of **linear** equations in variables a, b, and c.

$$\begin{cases} a+b+c=0 \end{cases}$$

2. [1pt] Write down an example of a system of equations in variables a, b, and c that is **not a linear system**.

$$\begin{cases} a^2+b^2+c^2=0 \end{cases}$$

3. [1pt] Write down an example of a system of **three linear equations** in its **echelon form** that contains **two free variables**.

$$\begin{cases} x \quad\quad\quad + u + w = 0 \\ \quad y \quad\quad + u + w = 0 \\ \quad\quad z + u + w = 0 \end{cases}$$

$\uparrow \; \uparrow$

free

4. [1pt] Write down an example of a 4×4 **nonsingular** matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. [1pt] Write down an example of a 4×4 **singular** matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[See Midterm1]

6. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

7. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that $\text{span}(S) = \mathbb{R}^3$ and S is not linearly independent.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

8. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) = \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

9. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

10. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a not subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

[See Midterm 2]

11. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not a homomorphism.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ x^2+y^2 \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

12. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is a homomorphism but not an isomorphism.

$$f(\vec{v}) = \vec{0} \quad \text{for all } \vec{v} \in \mathbb{R}^2.$$

13. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is an isomorphism.

$$f(\vec{v}) = \vec{v} \quad \text{for all } \vec{v} \in \mathbb{R}^2.$$

14. [1pt] Suppose V_1 and V_2 are two subspaces of $\mathbb{R}^3$. Give an example of V_1 and V_2 such that they are not linearly independent (in terms of subspaces).

$$V_1 = V_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

15. [1pt] Suppose V_1 and V_2 are two subspaces of $\mathbb{R}^3$. Give an example of V_1 and V_2 such that they are linearly independent (in terms of subspaces).

$$V_1 = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \quad V_2 = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

16. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2\}$ with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Define a homomorphism $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(\mathbf{v}_1) = 4\mathbf{u}_1$, $f(\mathbf{v}_2) = 6\mathbf{u}_2$, and $f(\mathbf{v}_3) = 8\mathbf{u}_1 + 8\mathbf{u}_2$.

(a) [2pt] Find $\mathrm{Rep}_{\mathcal{B},\mathcal{D}}(f)$.

$$f(\vec{v}_1) = 4\vec{u}_1 + 0\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$f(\vec{v}_2) = 0\vec{u}_1 + 6\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$f(\vec{v}_3) = 8\vec{u}_1 + 8\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix} 8 \\ 8 \end{pmatrix}.$$

$$\Rightarrow Rep_{B,D}(f) = \begin{pmatrix} 4 & 0 & 8 \\ 0 & 6 & 8 \end{pmatrix}.$$

(b) [3pt] Find a matrix A such that $f(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^3$.

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f(\vec{v}_1) = 4\vec{u}_1 = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f(\vec{v}_2 - \vec{v}_1) = 6\vec{u}_2 - 4\vec{u}_1 = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = f(\vec{v}_3 - \vec{v}_2) = 8\vec{u}_1 + 2\vec{u}_2 = \begin{pmatrix} 22 \\ 28 \end{pmatrix}.$$

$$\Rightarrow A = \begin{pmatrix} 8 & 10 & 22 \\ 12 & 0 & 28 \end{pmatrix}.$$

Check: $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \end{pmatrix} = 6\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 40 \\ 40 \end{pmatrix} = 8\vec{u}_1 + 8\vec{u}_2.$$

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17. [5pt] Let $f : V \to W$ be a homomorphism. Show that $f(X)$ is a subspace of W if X is a subspace of V.

See ver. A.

18. [5pt] Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if the null space of f is $\{\mathbf{0}\}$.

See ver. A.

19. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) [extra 1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

$$\operatorname{Rep}_B(M) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}.$$

$$A \cdot \operatorname{Rep}_B(M) = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 5 \\ 0 \\ 6 \end{pmatrix} \longrightarrow f(M) = 2E_{11} + 0E_{12} + 3E_{13} + 5E_{21} + 0E_{22} + 6E_{23} = \begin{pmatrix} 2 & 0 & 3 \\ 5 & 0 & 6 \end{pmatrix}.$$

(b) [extra 2pt] Find the range of f.

$$\text{Colspace}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\Longrightarrow \cancel{\text{span}}\ \text{range}(f) = \text{span}\{E_{11}, E_{13}, E_{21}, E_{23}\}.$$

(c) [extra 2pt] Find the nullspace of f.

$$\text{nullspace}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\Longrightarrow \text{nullspace}(f) = \text{span}\{E_{11}, E_{21}\}.$$

20. [extra 2pt] Recall that $\mathcal{L}(V, W)$ is the space of all homomorphisms from V to W. Let $V = \mathcal{M}_{4\times 5}$ be the space of all 4×5 real matrices. Let $W = \mathcal{P}_{100}$ be the space of all polynomials with real coefficients and of degree at most 100. Answer the following questions:

(a) What is the zero vector in $\mathcal{L}(V, W)$?

(b) What is the dimension of V?

(c) What is the dimension of W?

(d) What is the dimension of $\mathcal{L}(V, W)$?

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	30 (+7)	