國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第一次期中考 October 14, 2019 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**25 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Write down an example of a system of **linear** equations in variables x, y, and z.

$$\begin{cases} x+y+z=0 \\ x+y+z=1 \end{cases}$$

2. [1pt] Write down an example of a system of equations in variables x, y, and z that is **not a linear system**.

$$\begin{cases} x^2 = 1 \\ y+z=5 \end{cases}$$

3. [1pt] Write down an example of a system of **two linear equations** in its **echelon form** that contains **three free variables**.

$$\begin{cases} x \quad + z + u + v = 0 \\ \quad y + z + u + v = 0 \end{cases}.$$

4. [1pt] Write down an example of a 4×4 **singular** matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. [1pt] Write down an example of a 4×4 **nonsingular** matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ \sqrt{5} \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) [1pt] Find the length $|\mathbf{u}|$.

$$|u| = \sqrt{2^2 + (-3)^2 + \sqrt{5}^2 + 0^2}$$
$$= \sqrt{4 + 9 + 5 + 0} \quad = \underline{\sqrt{18}}$$

(b) [1pt] Find the length $|\mathbf{v}|$.

$$|v| = \sqrt{0^2 + 1^2 + 0^2 + 1^2}$$
$$= \underline{\sqrt{2}}$$

(c) [1pt] Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$u \cdot v = 2 \cdot 0 + (-3) \cdot 1 + \sqrt{5} \cdot 0 + 0 \cdot 1 = -3.$$

$$\Rightarrow \cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-3}{\sqrt{18} \cdot \sqrt{2}} = \frac{-3}{6} = -\frac{1}{2}$$

$$\Rightarrow \theta = \underline{\frac{2}{3}\pi}$$

(d) [2pt] Find a vector $\mathbf{w}$ such that the angle between $\mathbf{w}$ and $\mathbf{v}$ is $\frac{\pi}{4}$. [The answer is not unique. You only need to find one.]

for example, $w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

$$w \cdot v = 2 \qquad \Rightarrow \cos\theta = \frac{2}{2\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

$$|w||v| = 2 \cdot \sqrt{2}$$

7. [5pt] Find the general solution of the following linear system.

$$\begin{cases} w + \ x - \ 2y \qquad = -2 \\ 5w + 5x - 10y + \ z = -15 \\ 9w + 9x - 18y + 2z = -28 \end{cases}$$

That is, find $\mathbf{p}$ and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k$ such that

$$\{\mathbf{p} + c_1\boldsymbol{\beta}_1 + \cdots + c_k\boldsymbol{\beta}_k : c_1, \ldots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

$$\left(\begin{array}{cccc|c} 1 & 1 & \overset{-2}{\cancel{1}} & 0 & -2 \\ 5 & 5 & -10 & 1 & -15 \\ 9 & 9 & -18 & 2 & -28 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & \overset{-2}{\cancel{1}} & 0 & -2 \\ 0 & 0 & \overset{0}{\cancel{5}} & 1 & -5 \\ 0 & 0 & \overset{0}{\cancel{9}} & 2 & -10 \end{array}\right)$$

$$\begin{array}{cc} & \begin{array}{cccc} & \text{free} & \text{free} & \end{array} \\ \rightsquigarrow & \underbrace{\left(\begin{array}{cccc} 1 & 1 & -2 & 0 \\ & & & 1 \\ & & & \end{array}\right.}_{R} \underbrace{\left|\begin{array}{c} -2 \\ -5 \\ \ \end{array}\right)}_{\vec{r}} \end{array}$$

• Solve $R\vec{v} = \vec{r}$ for $\vec{p}$ by setting $x = y = 0$.

$$\Rightarrow \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & -2 \\ & & & 1 & -5 \\ & & & & \end{array}\right) \\ \begin{array}{cccc} & \uparrow & \uparrow & \\ & 0 & 0 & \end{array} \end{array} \Rightarrow \vec{p} = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 \\ 0 \\ 0 \\ -5 \end{pmatrix}}}$$

• Solve $R\vec{v} = \vec{0}$ for $\vec{\beta_1}, \vec{\beta_2}$.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ & & & 1 & 0 \\ & & & & \end{array}\right) \\ \begin{array}{cccc} & \uparrow & \uparrow & \\ & 1 & 0 & \end{array} \end{array} \Rightarrow \vec{\beta_1} = \underline{\underline{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}}$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 0 \\ & & & 1 & 0 \\ & & & & \end{array}\right) \\ \begin{array}{cccc} & \uparrow & \uparrow & \\ & 0 & 1 & \end{array} \end{array} \Rightarrow \vec{\beta_2} = \underline{\underline{\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}}$$

8. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -5 & 4 \\ -1 & -4 & 6 & -1 \\ 3 & 12 & -13 & 19 \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is known that $\mathbf{R}$ is the reduced echelon form of $\mathbf{A}$. Write the row $\begin{bmatrix} 1 & 4 & 0 & 0 \end{bmatrix}$ as a linear combination of rows of $\mathbf{A}$.

$$A \underset{-3\rho_1+\rho_3}{\overset{\rho_1+\rho_2}{\rightsquigarrow}} \begin{pmatrix} 1 & 4 & -5 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 7 \end{pmatrix} \overset{-2\rho_2+\rho_3}{\rightsquigarrow} \begin{pmatrix} 1 & 4 & -5 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{-4\rho_3+\rho_1}{\overset{-3\rho_3+\rho_2}{\rightsquigarrow}} \begin{pmatrix} 1 & 4 & -5 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \overset{5\rho_2+\rho_1}{\rightsquigarrow} \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & -4 \\ & 1 & -3 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -3 & & 1 \end{pmatrix} A = R.$$

$$\begin{pmatrix} 1 & 5 & -19 \\ & 1 & -3 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -5 & -2 & 1 \end{pmatrix} A = R.$$

$$\begin{pmatrix} 101 & 43 & -19 \\ 16 & 7 & -3 \\ -5 & -2 & 1 \end{pmatrix} A = R.$$

$$\Rightarrow 101(1 \ 4 \ -5 \ 4) + 43(-1 \ -4 \ 6 \ -1) + (-19)(3 \ 12 \ -13 \ 19) = (1 \ 4 \ 0 \ 0)$$

9. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 9 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

Is it possible to obtain **B** from **A** by some row operations? [This is a yes-or-no question, but you have to provide your ~~answer~~ reason.]

$$A \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 7 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ \cancel{1} & \cancel{2} & \cancel{3} \\ 2 & 3 & 3 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \cancel{0}1 & 0 & 3 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & -6 \\ & 1 & \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$B \; \cancel{0} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & \cancel{3}0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & \\ 0 & -3 & \\ & & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Same reduced echelon form $\Rightarrow$ Yes.

10. [extra 2pt] It is known that the following row operations are correct.

$$\begin{bmatrix}13\\23\end{bmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{bmatrix}13\\10\end{bmatrix} \xrightarrow{-\rho_2+\rho_1} \begin{bmatrix}3\\10\end{bmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{bmatrix}3\\1\end{bmatrix}$$

Find two integers a and b such that $a \cdot 13 + b \cdot 23 = 1$.

$$\begin{pmatrix}1 & \\ -3 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -1 & 1\end{pmatrix}\begin{pmatrix}13\\23\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\begin{pmatrix}1 & \\ -3 & 1\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}13\\23\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\\ -7 & 4\end{pmatrix}\begin{pmatrix}13\\23\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\Rightarrow \underset{a}{\underset{\uparrow}{(-7)}} \cdot 13 + \underset{b}{\underset{\uparrow}{(4)}} \cdot 23 = 1.$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	25 (+2)	