國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 November 25, 2019 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ______________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	7 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

2. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that $\text{span}(S) = \mathbb{R}^3$ and S is not linearly independent.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

3. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) = \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

5. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a not subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

6. [5pt] Find the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & -1 & 1 \\ -3 & -14 & 2 & -7 \\ 15 & 70 & -9 & 31 \\ -13 & -61 & 8 & -24 \end{bmatrix}.$$

$$(A|I) = \left(\begin{array}{cccc|cccc} 1 & 5 & -1 & 1 & 1 & & & \\ -3 & -14 & 2 & -7 & & 1 & & \\ 15 & 70 & -9 & 31 & & & 1 & \\ -13 & -61 & 8 & -24 & & & & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 5 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -4 & 3 & 1 & & \\ 0 & -5 & 6 & 16 & -15 & & 1 & \\ 0 & 4 & -5 & -11 & 13 & & & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & \cancel{5}\ 0 & \cancel{4}\ 4 & 21 & -14 & \cancel{0}\ -5 & 0 & 0 \\ 0 & 1 & -1 & -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 & 1 & 0 \\ 0 & 0 & \cancel{1}\ -1 & 5 & 1 & -4 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 37 & -14 & -25 & -4 & 0 \\ 0 & 1 & 0 & -8 & 3 & 6 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & & & & \cancel{23}\ -51 & -62 & -41 & -37 \\ & 1 & & & 11 & 14 & 9 & 8 \\ & & 1 & & \cancel{40}\ 4 & 9 & 5 & 4 \\ & & & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

So $$A^{-1} = \begin{pmatrix} -51 & -62 & -41 & -37 \\ 11 & 14 & 9 & 8 \\ 4 & 9 & 5 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

7. [5pt] Let

$$V = \text{span}\left(\left\{\begin{bmatrix}1\\-5\\4\end{bmatrix}, \begin{bmatrix}-1\\5\\-4\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\20\\-16\end{bmatrix}, \begin{bmatrix}1\\-4\\4\end{bmatrix}\right\}\right).$$

Find **a basis** and **the dimension** of V.

$$\begin{pmatrix} 1 & -1 & 0 & -4 & 1 \\ -5 & 5 & 0 & 20 & -4 \\ 4 & -4 & 0 & -16 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{array}{ccccc} \uparrow & & & & \uparrow \\ \text{leading} & & & & \text{leading.} \end{array}$$

$\Rightarrow$ basis $= \left\{ \begin{pmatrix}1\\-5\\4\end{pmatrix}, \begin{pmatrix}1\\-4\\4\end{pmatrix} \right\}.$

dim $= 2$.

8. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & -4 & 1 \\ -5 & 5 & 0 & 20 & -4 \\ 4 & -4 & 0 & -16 & 4 \end{bmatrix}.$$

(a) [2pt] Find **a basis** and **the dimension** of the row space of $\boldsymbol{A}$.

See the ref in Q7.

basis $= \{(1\ -1\ 0\ -4\ 1),\ (0\ 0\ 0\ 0\ 1)\}$.

dim $= 2$.

(b) [3pt] Find **a basis** and **the dimension** of the null space of $\boldsymbol{A}$.

Solve for $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$.

$$\begin{pmatrix} 1 & -1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\uparrow\ \uparrow\ \uparrow$

$1\ \ 0\ \ 0$

basis $= \{\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\}$,

dim $= 3$

$$\begin{pmatrix} 1 & -1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\uparrow\ \uparrow\ \uparrow$

$0\ \ 1\ \ 0$

$$\begin{pmatrix} 1 & -1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_3 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\uparrow\ \uparrow\ \uparrow$

$0\ \ 0\ \ 1$

9. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} ? & ? & ? & ? & a_5 \\ ? & a_2 & ? & ? & 0 \\ ? & 0 & ? & a_4 & 0 \\ ? & 0 & a_3 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a 5×5 real matrix such that $a_1, \ldots, a_5$ are nonzero and each question mark represents an unknown value. Show that the columns of $\mathbf{A}$ form a linearly independent set.

Let $\vec{v}_1, \ldots, \vec{v}_5$ be the columns of A.

Suppose $c_1\vec{v}_1 + \ldots + c_5\vec{v}_5 = \vec{0}$.

$$\Rightarrow A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

5th row: $(a_1 \ 0 \ 0 \ 0 \ 0) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0 \Rightarrow c_1 = 0$ $\cancel{c_5 = 0}$.

4th row: $(? \ 0 \ a_3 \ 0 \ 0) \begin{pmatrix} 0 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0 \Rightarrow c_3 = 0$

3rd row: $(? \ 0 \ ? \ a_4 \ 0) \begin{pmatrix} 0 \\ c_2 \\ 0 \\ c_4 \\ c_5 \end{pmatrix} \Rightarrow c_4 = 0.$

Similarly, 2nd row $\Rightarrow c_2 = 0$

1st row $\Rightarrow c_5 = 0$

So $c_1 = \ldots = c_5 = 0 \Rightarrow$ the columns form a linearly independent set.

10. [5pt] Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a basis of some vector space V. Let $\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{y}_2 = \mathbf{x}_1 - \mathbf{x}_2$. $y_3 = x_3$. Show that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is also a basis of V.

Claim: $\operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = V$.

Since $\vec{y}_1, \vec{y}_2, \vec{y}_3 \in \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

$\Rightarrow \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} \subseteq \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

Since $\vec{x}_1 = (\vec{y}_1 + \vec{y}_2)/2$, $\vec{x}_2 = (\vec{y}_1 - \vec{y}_2)/2$, $\vec{x}_3 = \vec{y}_3$ $\in \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

$\Rightarrow \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subseteq \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

~~Clai~~

Thus, $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ is a spanning set of V with 3 vectors, and $\dim V = 3$.

So $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ is a basis.

11. [extra 2pt] Let

$$f_1 = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)},$$
$$f_2 = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)},$$
$$f_3 = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}, \text{ and}$$
$$f_4 = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

be four polynomials. Show that any polynomial f of degree at most 3 can be written as a linear combination of $f_1, \ldots, f_4$. That is, for a given

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

find coefficients $c_1, \ldots, c_4 \in \mathbb{R}$ such that

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4.$$

Let ~~B~~ P_3 be the space of all polynomials of degree ≤ 3.

~~dim B~~ $\dim P_3 = 4$.

Claim: $\{f_1, f_2, f_3, f_4\}$ is indep

If $k_1 f_1 + k_2 f_2 + k_3 f_3 + k_4 f_4 = 0$.

then $k_1 \underbrace{f_1(1)}_{1} + k_2 \underbrace{f(1)}_{0} + k_3 \underbrace{f_3(1)}_{0} + k_4 \underbrace{f_4(1)}_{0.} = 0 \Rightarrow k_1 = 0$.

Similarly, let $x = 2, 3, 4 \Rightarrow k_2 = k_3 = k_4 = 0$.

$\Rightarrow \{f_1, f_2, f_3, f_4\}$ is a basis of P_3.

$\Rightarrow$ Any $f \in P_3$ is a linear combination of $\{f_1, f_2, f_3, f_4\}$.

In fact, $c_1 = f(1)$, $c_2 = f(2)$, $c_3 = f(3)$, $c_4 = f(4)$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	