國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第二次期中考 November 25, 2019 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	7 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that $\text{span}(S) = \mathbb{R}^3$ and S is not linearly independent.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

2. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

3. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) = \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

4. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a not subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

5. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

6. [5pt] Find the inverse of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 0 & -1 \\ -3 & -14 & 3 & 1 \\ 10 & 48 & -5 & -4 \\ 37 & 178 & -18 & -16 \end{bmatrix}.$$

$$(A \mid I) = \left(\begin{array}{cccc|cccc} 1 & 5 & 0 & -1 & 1 & & & \\ -3 & -14 & 3 & 1 & & 1 & & \\ 10 & 48 & -5 & -4 & & & 1 & \\ 37 & 178 & -18 & -16 & & & & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 5 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 3 & 1 & & \\ 0 & -2 & -5 & 6 & -10 & & 1 & \\ 0 & -7 & -18 & 21 & -37 & & & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & -15 & 9 & -14 & -5 & 0 & 0 \\ 0 & 1 & 3 & -2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -4 & 2 & 1 & 0 \\ 0 & 0 & 3 & 7 & -16 & 7 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 39 & -74 & 25 & 15 & 0 \\ 0 & 1 & 0 & -8 & 15 & -5 & -3 & 0 \\ 0 & 0 & 1 & 2 & -4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 1 & -3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & & & & 82 & -14 & 132 & -39 \\ & 1 & & & -17 & 3 & -27 & 8 \\ & & 1 & & 4 & 0 & 7 & -2 \\ & & & 1 & -4 & 1 & -3 & 1 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} 82 & -14 & 132 & -39 \\ -17 & 3 & -27 & 8 \\ 4 & 0 & 7 & -2 \\ -4 & 1 & -3 & 1 \end{pmatrix}.$

7. [5pt] Let

$$V = \operatorname{span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}3\\9\\15\end{bmatrix}, \begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}5\\16\\25\end{bmatrix}, \begin{bmatrix}-23\\-73\\-115\end{bmatrix}\right\}\right).$$

Find **a basis** and **the dimension** of V.

$$\begin{pmatrix} 1 & 3 & 1 & 5 & -23 \\ 3 & 9 & 3 & 16 & -73 \\ 5 & 15 & 5 & 25 & -115 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 1 & 5 & -23 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

leading.

$$\text{Basis} = \left\{ \begin{pmatrix}1\\3\\5\end{pmatrix}, \begin{pmatrix}5\\16\\25\end{pmatrix} \right\}$$

$$\dim = 2.$$

8. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 & 5 & -23 \\ 3 & 9 & 3 & 16 & -73 \\ 5 & 15 & 5 & 25 & -115 \end{bmatrix}.$$

(a) [2pt] Find **a basis** and **the dimension** of the row space of $\boldsymbol{A}$.

see the ref in Q7.

basis $= \{(1\ 3\ 1\ 5\ -23), (0\ 0\ 0\ 1\ -4)\}$.

dim $= 2$.

(b) [3pt] Find **a basis** and **the dimension** of the null space of $\boldsymbol{A}$.

~~ba~~ Find $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$.

$$\begin{pmatrix} 1 & 3 & 1 & 5 & -23 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & \uparrow & \uparrow & & \uparrow \\ & 1 & 0 & & 0 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 & -23 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & \uparrow & \uparrow & & \uparrow \\ & 0 & 1 & & 0 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 & -23 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{\beta}_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccccc} & \uparrow & \uparrow & & \uparrow \\ & 0 & 0 & & 1 \end{array}$$

basis $= \{\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\}$.

dim $= 3$.

9. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} ? & ? & ? & ? & a_5 \\ ? & a_2 & ? & ? & 0 \\ ? & 0 & a_3 & ? & 0 \\ a_1 & 0 & 0 & ? & 0 \\ 0 & 0 & 0 & a_4 & 0 \end{bmatrix}$$

be a 5×5 real matrix such that $a_1, \ldots, a_5$ are nonzero and each question mark represents an unknown value. Show that the columns of $\mathbf{A}$ form a linearly independent set.

Let $\vec{v_1}, \ldots, \vec{v_5}$ be the columns of A.

Suppose $c_1\vec{v_1} + \ldots + c_5\vec{v_5} = \vec{0}$

5th entry $\Rightarrow$ $c_4 = 0$

4~~st~~th entry $\Rightarrow$ $c_1 = 0$

3rd entry $\Rightarrow$ $c_3 = 0$ $\Rightarrow$ $c_1 = \ldots = c_5 = 0$

2nd entry $\Rightarrow$ $c_2 = 0$

1st entry $\Rightarrow$ $c_5 = 0$

So columns of A form a linearly indep set.

10. [5pt] Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a basis of some vector space V. Let $\mathbf{y}_1 = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{y}_2 = \mathbf{x}_1 + \mathbf{x}_2$, $\vec{y}_3 = \vec{x}_3$. Show that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is also a basis of V.

Claim: $\text{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = V$.

$$\left.\begin{aligned} \vec{y}_1 &= \vec{x}_1 - \vec{x}_2 \\ y_2 &= \vec{x}_1 + \vec{x}_2 \\ \vec{y}_3 &= \vec{x}_3 \end{aligned}\right\} \in \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}.$$

$\Rightarrow \text{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} \leq \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

$$\left.\begin{aligned} \vec{x}_1 &= (\vec{y}_1 + \vec{y}_2)/2 \\ \vec{x}_2 &= (\vec{y}_2 - \vec{y}_1)/2 \\ \vec{x}_3 &= \vec{y}_3 \end{aligned}\right\} \in \text{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}.$$

$\Rightarrow \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subseteq \text{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

So $\vec{y}_1, \vec{y}_2, \vec{y}_3$ span V and $\dim V = 3$.

$\Rightarrow \{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ is a basis.

11. [extra 2pt] Let

$$f_1 = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)},$$
$$f_2 = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)},$$
$$f_3 = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}, \text{ and}$$
$$f_4 = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

be four polynomials. Show that any polynomial f of degree at most 3 can be written as a linear combination of $f_1, \ldots, f_4$. That is, for a given

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

find coefficients $c_1, \ldots, c_4 \in \mathbb{R}$ such that

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4.$$

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	