Sample Solutions for Sample Questions 10

1. $$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 12 & 5 \\ 0 & 0 & 11 & 6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 11 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row space :

basis

$$\langle (1,0,-4,-1),\ (0,1,1,-1),\ (0,0,11,6) \rangle$$

dimension = 3 ⟵ row rank.

Column space :

basis $$\left\langle \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\rangle$$

dimension = 3 ⟵ column rank.

Null space :

basis $$\left\langle \begin{pmatrix} -13/11 \\ 17/11 \\ -6/11 \\ 1 \end{pmatrix} \right\rangle$$

(by setting $w\,\cancel{z}=1$ and solve $A\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0$.)

dimension = 1 ⟵ nullity

2.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -5 & 3 & -4 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -5 & 3 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 8 & 16 \end{pmatrix}.$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ & 1 & 1 & 4 \\ & & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 & -10 \\ & 1 & 1 & 4 \\ & & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ & 1 & 0 & 2 \\ & & 1 & 2 \end{pmatrix}$$

Row space :

basis : $\langle (1,0,0,-2), (0,1,0,2), (0,0,1,2) \rangle$

dimension $= 3 \leftarrow$ rank

Column space :

basis : $\left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

dimension $= 3 \leftarrow$ rank.

Null space :

basis : $\left\langle \begin{pmatrix} 2 \\ -2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

dimension $= 1 \leftarrow$ nullity.

3.

Since $a \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ 0 & d - b \cdot \frac{c}{a} \end{pmatrix}.$$

To have rank $= 1$,

$$d - b \cdot \frac{c}{a} = 0 \Rightarrow d = b \cdot \frac{c}{a}.$$

4. Claim: consistent $\Leftrightarrow$ $\text{rank}(A) = \text{rank}([A|b])$.

Proof 1:

Suppose R is the reduced echelon form of A.

Then $[R \mid r]$ is the reduced echelon form of $[A \mid b]$

for some column vector r.

rank (A)

consistent means # of nonzero rows in R

= # of nonzero rows in ~~A~~ $[R \mid r]$

rank ([A | b]).

Proof 2:

$Ax = b$ is consistent $\Leftrightarrow$ $b \in \text{Columnspace}(A)$

$\Leftrightarrow$ ~~So~~ $\text{Colspace}(A) = \text{Colspace}([A|b])$.

$\Leftrightarrow$ $\dim \text{Colspace}(A) = \dim \text{Colspace}([A|b])$

$\Leftrightarrow$ $\text{rank}(A) = \text{rank}([A|b])$

5.

(a).

$Ax=b$ is consistent for all $b \in \mathbb{R}^m$

$\Leftrightarrow b \in \text{Colspace}(A)$ for all $b \in \mathbb{R}^m$

$\Leftrightarrow \text{Colspace}(A) = \mathbb{R}^m$

$\Leftrightarrow$ ~~ra~~ $\dim \text{Colspace}(A) = m$

$\Leftrightarrow \text{rank}(A) = m.$ ← # of rows.

(b). $Ax=b$ has a unique solution for all $b \in \mathbb{R}^m$ making $Ax=b$ consistent.

$\Leftrightarrow Ax=b$ has a unique solution for all $b \in \text{Colspace}(A)$.

$\Leftrightarrow$ every $b \in \text{Colspace}(A)$ has a unique representation with respect to columns in A.

$\Leftrightarrow$ columns in A form a linearly indep. set.

$\Leftrightarrow$ ~~Co~~ $\dim \text{Colspace}(A) = n.$

$\Leftrightarrow \text{rank}(A) = n$ ← # of columns.

6.

① Let S_A and S_B be bases of Colspace(A) and Colspace(B), respectively.

Let $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$, $B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}$.

Then $A+B = \begin{bmatrix} | & | & & | \\ a_1+b_1 & a_2+b_2 & \cdots & a_n+b_n \\ | & | & & | \end{bmatrix}$.

So $a_i + b_i = 1 \cdot a_i + 1 \cdot b_i \in \text{span}(S_A \cup S_B)$.

Therefore, $\text{Colspace}(A+B) \subseteq \text{span}(S_A \cup S_B)$.

$$\Rightarrow \text{rank}(A+B) = \dim \text{Colspace}(A+B)$$
$$\leq \dim \text{span}(S_A \cup S_B)$$
$$\leq |S_A \cup S_B| = \text{rank}(A) + \text{rank}(B).$$

②

Each column of AB is a linear combination of columns of A $\Rightarrow$ $\text{Colspace}(AB) \subseteq \text{Colspace}(A)$

$\Rightarrow$ $\text{rank}(AB) \leq \text{rank}(A)$.

Similarly, each ~~column~~ row of AB is a linearly combination of rows in B $\Rightarrow$ $\text{Rowspace}(AB) \subseteq \text{Rowspace}(B)$

$\Rightarrow$ $\text{rank}(AB) \leq \text{rank}(B)$.

7. Let $B = \langle \vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_n \rangle$ be a basis of V.

Let $D = \langle \vec{\delta}_1, \vec{\delta}_2, \dots, \vec{\delta}_m \rangle$ be a linearly indep. set.

Since B is a basis, B is indep. and $\text{span}(B) = V$.

不一定是 $\vec{\beta}_1$

① $\vec{\delta}_1 \in \text{span}(B) \Rightarrow$ there is $\hat{\beta}_1 \in B$ such that

~~$B_1 = \langle \vec{\delta}_1\ B \cup \{\vec{\delta}_1\} \setminus \{\vec{\beta}_1$~~

$B_1 = B \cup \{\vec{\delta}_1\} \setminus \{\hat{\beta}_1\}$ is a basis,

by Exchange Lemma.

② $\vec{\delta}_2 \in \text{span}(B_1) \Rightarrow$ there is $\hat{\beta}_2 \in B$ such that
($\because B_1$ is a basis)

$B_2 = B_1 \cup \{\vec{\delta}_2\} \setminus \{\hat{\beta}_2\}$ is a basis.

⋮

ⓚ $\vec{\delta}_k \in \text{span}(B_{k-1}) \Rightarrow \vec{\delta}_k = c_1\vec{\delta}_1 + \dots + c_{k-1}\vec{\delta}_{k-1} + \vec{b}$ ~~linear combinat~~
($\because B_{k-1}$ is a basis) where $\vec{b}$ is a linear combination of the $\vec{\beta}_i$'s in B_{k-1}.

Since $\{\vec{\delta}_1, \dots, \vec{\delta}_k\}$ is indep., $\vec{b} \neq 0$.

$\Rightarrow$ there is $\hat{\beta}_k \in B$ such that

$B_k = B_{k-1} \cup \{\vec{\delta}_k\} \setminus \{\hat{\beta}_k\}$ is a basis.

⋮

ⓜ B_m is a basis ~~but~~ of the form

$\langle \underbrace{\vec{\delta}_1, \dots, \vec{\delta}_m}_{D}, \underbrace{n-m \text{ vectors in } B}_{B'} \rangle$.

$= D \cup B'$.

Thus, this completes the proof.