# Sample Solutions for Sample Questions 11.

1. Suppose $\vec{x} \in \mathbb{R}^3$ has the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

We want to write

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} k \\ k \\ k \end{pmatrix}}_{\in V_1} + \underbrace{\begin{pmatrix} a-k \\ b-k \\ c-k \end{pmatrix}}_{\in V_2}.$$

$\Rightarrow \quad (a-k) + (b-k) + (c-k) = 0$

since $\begin{pmatrix} a-k \\ b-k \\ c-k \end{pmatrix}$ 内積 $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$.

$\Rightarrow \quad k = \frac{1}{3}(a+b+c)$.

$\Rightarrow \quad \vec{x}_1 = \begin{pmatrix} k \\ k \\ k \end{pmatrix}, \ \vec{x}_2 = \begin{pmatrix} a-k \\ b-k \\ c-k \end{pmatrix}, \quad k = \frac{1}{3}(a+b+c).$

2. $\{V_1, V_2\}$ is not linearly indep

since $\vec{V}_1 + \vec{V}_3 \in V_1$, $\vec{V}_2 + \vec{V}_4 \in V_2$ but $\underset{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\underset{\uparrow}{(\vec{V}_1 + \vec{V}_3)}} - \underset{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\underset{\uparrow}{(\vec{V}_2 + \vec{V}_4)}} = \vec{0}$.

3. $V_1 + V_2 = \operatorname{span}\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4\}$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow \uparrow \uparrow$
leading

so $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is a basis of $V_1 + V_2$.

4.

(a) Not an isomorphism

because $f$ is not one-to-one.

e.g. $f\begin{pmatrix}1\\0\\0\end{pmatrix} = f\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

(b) Yes.

"one-to-one" and "onto"

Define $g\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}w & z-w\\ y-z & x-y\end{pmatrix}$.

Note that $g\left(f\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right) = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

and $f\left(g\begin{pmatrix}x\\y\\z\\w\end{pmatrix}\right) = \begin{pmatrix}x\\y\\z\\w\end{pmatrix}$

$\Rightarrow$ $f$ is a bijection since the inverse $f^{-1}$ exists.

"preserve structure"

$$f\left(\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} + \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\right) = f\begin{pmatrix}a_1+a_2 & b_1+b_2\\ c_1+c_2 & d_1+d_2\end{pmatrix}$$

$$= \begin{pmatrix}a_1+a_2+b_1+b_2+c_1+c_2+d_1+d_2\\ a_1+a_2+b_1+b_2+c_1+c_2\\ a_1+a_2+b_1+b_2\\ a_1+a_2\end{pmatrix} = \begin{pmatrix}a_1+b_1+c_1+d_1\\ a_1+b_1+c_1\\ a_1+b_1\\ a_1\end{pmatrix} + \begin{pmatrix}a_2+b_2+c_2+d_2\\ a_2+b_2+c_2\\ a_2+b_2\\ a_2\end{pmatrix}$$

$$= f\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix} + f\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

[4(b) continued].

$$f\left(r\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = f\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = \begin{pmatrix} ra+rb+rc+rd \\ ra+rb+rc \\ ra+rb \\ ra \end{pmatrix}$$

$$= r\begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{pmatrix} = r\cdot f\begin{pmatrix} a & b \\ c & d\end{pmatrix}.$$

(c) No, f doesn't preserve structure

e.g. $\left.\begin{array}{l} f(1+2) = 3^3 = 27 \\ f(1)+f(2) = 1^3+2^3 = 9 \end{array}\right\}$ 不同

5. Let $B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ be a basis of $V$.

Let $f: V \longrightarrow \mathbb{R}^2$, $\vec{v} \longmapsto Rep_B(\vec{v})$. That is, $f\left(a\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\vec{\beta_1}} + b\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{\beta_2}}\right) = \begin{pmatrix} a \\ b \end{pmatrix}$.

"one-to-one"

If $f(\vec{v_1}) = f(\vec{v_2}) = \begin{pmatrix} a \\ b \end{pmatrix}$, then $\vec{v_1} = \vec{v_2} = a\vec{\beta_1} + b\vec{\beta_2}$

"onto"

Any vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ is mapped by $f(a\vec{\beta_1} + b\vec{\beta_2}) = \begin{pmatrix} a \\ b \end{pmatrix}$.

"preserve structure"

$\cancel{f(a_1\vec{v_1}+b_1\vec{v_2}}$ $f\left((a_1\vec{\beta_1}+b_1\vec{\beta_2}) + (a_2\vec{\beta_1}+b_2\vec{\beta_2})\right) = f\left((a_1+a_2)\vec{\beta_1} + (b_1+b_2)\vec{\beta_2}\right)$

$$= \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = f(a_1\vec{\beta_1}+b_1\vec{\beta_2}) + f(a_2\vec{\beta_1}+b_2\vec{\beta_2})$$

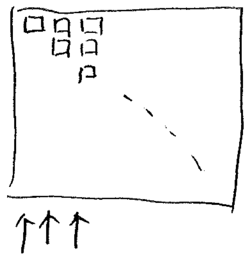
$$f\left(r(a\vec{\beta_1}+b\vec{\beta_2})\right) = f(ra\vec{\beta_1} + rb\vec{\beta_2})$$

$$= \begin{pmatrix} ra \\ rb \end{pmatrix} = r\begin{pmatrix} a \\ b \end{pmatrix} = r\cdot f(a\vec{\beta_1} + b\vec{\beta_2}).$$

6. Simply count the dimension.

$\dim M_{m\times n} = mn$, so $a = mn$

$\dim \cancel{S_m}\, S_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$, so $b = \frac{n(n+1)}{2}$.



$1+2+3+\cdots+n$ 對稱矩陣能自由改變的只有這些.

7. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$.

~~Then~~ Suppose $\vec{y} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

Then $\vec{y} = A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = A \cdot \mathrm{Rep}_B(\vec{y})$.

Since $A$ is a square matrix with $\operatorname{rank} A = n$,

$A$ has inverse.

Take $B = A^{-1}$.

Then $\mathrm{Rep}_B(\vec{y}) = B\vec{y}$.