Sample Questions 13

1. Find a basis for the vector space V.
   (a) $V = \mathbb{R}^3$
   (b) $V = \mathcal{P}_3$, the space of all polynomials of degree at most 3
   (c) $V = \mathcal{M}_{3\times 3}$, the space of all 3×3 matrices
   (d) $V = \mathcal{S}_3$, the space of all 3×3 symmetric matrices ($\mathbf{A} = \mathbf{A}^\top$)
   (e) $V = \mathcal{K}_3$, the space of all 3×3 skew-symmetric matrices ($\mathbf{A} = -\mathbf{A}^\top$)

2. With the given basis $\mathcal{B}$ and the representation, find the vector $\mathbf{v}$.
   (a) $\mathcal{B} = \{1, x-1, (x-1)^2\}$ and $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$
   (b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ and $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$
   (c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v}) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

3. Let $\mathcal{B}$ be as Problem 2(a), (b), and (c), respectively. Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{v})$.
   (a) $\mathbf{v} = x^2$
   (b) $\mathbf{v} = \begin{bmatrix} 5 & 7 \\ 7 & 3 \end{bmatrix}$
   (c) $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

4. Suppose f is a homomorphism with

$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ and } f(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$$

   Find $f(\begin{bmatrix} 2 \\ 4 \end{bmatrix})$.

5. Suppose $\mathbf{A}$ is a 2×3 matrix with $\mathbf{A}\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{A}\mathbf{e}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\mathbf{A}\mathbf{e}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. Find $\mathbf{A}$.

6. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^3$. Find $\mathrm{Rep}_{\mathcal{B}}(\mathbf{e}_1)$, $\mathrm{Rep}_{\mathcal{B}}(\mathbf{e}_2)$, and $\mathrm{Rep}_{\mathcal{B}}(\mathbf{e}_3)$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$.

7. Suppose f is a homomorphism with

$$f(\begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}) = f(\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}) = f(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

   Find a matrix $\mathbf{A}$ such that $f(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.