Sample Solutions for Sample Questions 13

1. There are many possible bases. The ones provided here are the "standard" ones.
(kind of)

(a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(b) $\{ 1, x, x^2, x^3 \}$

(c) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$

$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

(d) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \right.$

$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

(e) $\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$.

2.

(a). $\vec{v} = 3 \cdot 1 + 4 \cdot (x-1) + 5 \cdot (x-1)^2$

【除非題目需要才要展開】

(b) $\vec{v} = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 3 & 5 \\ 5 & 4 \end{pmatrix}$

(c) $\vec{v} = 3 \cdot \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 19 \\ 55 \end{pmatrix}$

3. (a) Find c_1, c_2, c_3 such that

$x^2 = c_1 \cdot 1 + c_2 \cdot (x-1) + c_3 (x-1)^2$.

$= c_1 \cdot 1$

$+ (-c_2) \cdot 1 + c_2 \cdot x$

$+ c_3 \cdot 1 + (-2c_3) x + c_3 x^2$

$\Rightarrow \begin{cases} c_1 - c_2 + c_3 = 0 \\ c_2 - 2c_3 = 0 \\ c_3 = 1 \end{cases} \Rightarrow Rep_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

3 (b).

$$\begin{pmatrix} 5 & 7 \\ 7 & 3 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 7\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \mathrm{Rep}_{\mathbb{B}}(\vec{v}) = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix}.$$

(c) Solve $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

$$\Rightarrow \mathrm{Rep}_{\mathbb{B}}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}$$

4. Find a representation of $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ by $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

$$\Rightarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So $f\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$= f\left(3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 3 \cdot f\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) f\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 3 \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} + (-1)\begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

5. Suppose $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.

$$A e_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Similarly, $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

6. By 3(c), $Rep_{\mathcal{B}}(\vec{e}_1) = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}$.

Solve $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\Rightarrow c_1 = 0$, $c_2 = 1$, $c_3 = -5$.

$\Rightarrow Rep_{\mathcal{B}}(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$

Solve $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow c_1 = c_2 = 0$, $c_3 = 1$.

$\Rightarrow Rep_{\mathcal{B}}(\vec{e}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

7. Find $f(\vec{e}_1)$, $f(\vec{e}_2)$, and $f(\vec{e}_3)$.

Then $A = \begin{pmatrix} | & | & | \\ f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \end{pmatrix}$.

By Problem 6,

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\5\\10\end{pmatrix} - 5\begin{pmatrix}0\\1\\5\end{pmatrix} + 15\begin{pmatrix}0\\0\\1\end{pmatrix} \xmapsto{f} 1\begin{pmatrix}3\\4\end{pmatrix} - 5\begin{pmatrix}3\\4\end{pmatrix} + 15\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \begin{pmatrix}33\\44\end{pmatrix}.$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\5\\10\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\5\end{pmatrix} - 5\begin{pmatrix}0\\0\\1\end{pmatrix} \xmapsto{f} 0\begin{pmatrix}3\\4\end{pmatrix} + 1\cdot\begin{pmatrix}3\\4\end{pmatrix} - 5\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \begin{pmatrix}-12\\-16\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\begin{pmatrix}1\\5\\10\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\\5\end{pmatrix} + 1\begin{pmatrix}0\\0\\1\end{pmatrix} \xmapsto{f} = 0\begin{pmatrix}3\\4\end{pmatrix} + 0\begin{pmatrix}3\\4\end{pmatrix} + 1\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \begin{pmatrix}3\\4\end{pmatrix}.$$

$$\Rightarrow A = \begin{pmatrix}33 & -12 & 3\\44 & -16 & 4\end{pmatrix}.$$

You may double-check that

$$A\begin{pmatrix}1\\5\\10\end{pmatrix} = A\begin{pmatrix}0\\1\\5\end{pmatrix} = A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$