

Sample Solutions for Sample Questions 14

1. Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $A = \begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \\ | & | & | \end{pmatrix}$

$$\vec{e}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_1) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 + 0 \cdot \vec{u}_3 = \vec{u}_1$$

$$\vec{e}_2 = -10 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_2) = -10 \cdot \vec{u}_1 + 1 \cdot \vec{u}_2 + 0 \cdot \vec{u}_3 = -9 \vec{u}_1 + \vec{u}_2$$

$$\vec{e}_3 = 95 \vec{v}_1 - 10 \cdot \vec{v}_2 + 1 \cdot \vec{v}_3$$

$$\Rightarrow f(\vec{e}_3) = 95 \vec{u}_1 - 10 \vec{u}_2 + 1 \cdot \vec{u}_3 = 86 \vec{u}_1 - 9 \vec{u}_2 + \vec{u}_3$$

So $A = \begin{pmatrix} | & | & | \\ \vec{u}_1 & -9\vec{u}_1 + \vec{u}_2 & 86\vec{u}_1 - 9\vec{u}_2 + \vec{u}_3 \\ | & | & | \end{pmatrix}$

$$= \begin{pmatrix} 3 & -27 & 258 \\ 4 & -36 & 344 \end{pmatrix}$$

2. Let $S_3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

$$\text{Then } \text{Rep}_{S_3, S_2}(f) = \begin{pmatrix} \text{Rep}_{S_2}(f(\vec{e}_1)) & \text{Rep}_{S_2}(f(\vec{e}_2)) & \text{Rep}_{S_2}(f(\vec{e}_3)) \\ | & | & | \\ | & | & | \end{pmatrix}$$

Note that $\text{Rep}_{S_2}(\vec{w}) = \vec{w}$ for all $\vec{w} \in \mathbb{R}^2$.

$$\text{So } \text{Rep}_{S_3, S_2}(f) = \begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \\ | & | & | \\ | & | & | \end{pmatrix}$$

= A in Problem 1.

(This is not a coincidence.)

3. Let $S_3 = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$

$$\text{Then } \text{Rep}_{S_3, D}(f) = \begin{pmatrix} \text{Rep}_D(f(\vec{e}_1)) & \text{Rep}_D(f(\vec{e}_2)) & \text{Rep}_D(f(\vec{e}_3)) \\ | & | & | \\ | & | & | \\ | & | & | \end{pmatrix}$$

We already computed $f(\vec{e}_1), f(\vec{e}_2), f(\vec{e}_3)$
in Problem 1.

$$f(\vec{e}_1) = \vec{u}_1 + 0\vec{u}_2 \Rightarrow \text{Rep}_D(f(\vec{e}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(\vec{e}_2) = -9\vec{u}_1 + 0\vec{u}_3 \Rightarrow \text{Rep}_D(f(\vec{e}_2)) = \begin{pmatrix} -9 \\ 0 \end{pmatrix}$$

$$f(\vec{e}_3) = 86\vec{u}_1 + 0\vec{u}_2 \Rightarrow \text{Rep}_D(f(\vec{e}_3)) = \begin{pmatrix} 86 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{S_3, D}(f) = \begin{pmatrix} 1 & -9 & 86 \\ 0 & 0 & 0 \end{pmatrix}$$

$$4. B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$$

$$\text{Then } \text{Rep}_{B, S_2}(f) = \begin{pmatrix} | & | & | \\ \text{Rep}_{S_2}(f(\vec{v}_1)) & \text{Rep}_{S_2}(f(\vec{v}_2)) & \text{Rep}_{S_2}(f(\vec{v}_3)) \\ | & | & | \end{pmatrix}$$

Again, $\text{Rep}_{S_2}(\vec{w}) = \vec{w}$ for all $\vec{w} \in \mathbb{R}^2$.

$$\text{Rep}_{B, S_2}(f) = \begin{pmatrix} | & | & | \\ f(\vec{v}_1) & f(\vec{v}_2) & f(\vec{v}_3) \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_1 & \vec{u}_1 \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \end{pmatrix}$$

$$5. \quad B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$$

$$\text{Then } \text{Rep}_{B,D}(f) = \begin{pmatrix} | & | & | \\ \text{Rep}_D(f(\vec{v}_1)) & \text{Rep}_D(f(\vec{v}_2)) & \text{Rep}_D(f(\vec{v}_3)) \\ | & | & | \end{pmatrix}$$

It's super easy.

$$f(\vec{v}_1) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \text{Rep}_D(f(\vec{v}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(\vec{v}_2) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \text{Rep}_D(f(\vec{v}_2)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(\vec{v}_3) = 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \Rightarrow \text{Rep}_D(f(\vec{v}_3)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So

$$\text{Rep}_{B,D}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

6.

$$\left[\begin{array}{l} \text{By Problem 5, } B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \text{You may check } B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } B \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \end{array} \right]$$

$$B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathbb{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{D}} \text{ means}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathbb{B}} \xrightarrow{B \cdot \square} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbb{D}}$$

$$1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 \xrightarrow{f} 1 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2 \quad \text{and}$$

$$\text{That is, } f(\vec{v}_1) = \vec{u}_1.$$

$$B \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathbb{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathbb{D}} \text{ means}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{\mathbb{B}} \xrightarrow{B \cdot \square} \begin{pmatrix} 2 \\ 0 \end{pmatrix}_{\mathbb{D}}$$

$$1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 \xrightarrow{f} 2 \cdot \vec{u}_1 + 0 \cdot \vec{u}_2$$

$$\text{That is, } f(\vec{v}_1 + \vec{v}_2) = 2\vec{u}_1.$$

7. Use $A = \begin{pmatrix} 3 & -27 & 258 \\ 4 & -36 & 344 \end{pmatrix}$ in Problem 1.

$$\begin{aligned} \text{range}(f) &\stackrel{\text{def}}{=} \{ f(\vec{v}) \mid \vec{v} \in \mathbb{R}^3 \} \\ &= \{ A\vec{v} \mid \vec{v} \in \mathbb{R}^3 \} \\ &= \text{Columnspace}(A) \\ &= \text{span} \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\} \longrightarrow \text{rank} = 1. \end{aligned}$$

$$\begin{aligned} \text{nullspace}(f) &\stackrel{\text{def}}{=} \{ \vec{v} \in \mathbb{R}^3 \mid f(\vec{v}) = \vec{0} \in \mathbb{R}^2 \} \\ &= \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{0} \in \mathbb{R}^2 \} \\ &= \text{nullspace}(A) \\ &= \text{span} \left\{ \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -86 \\ 0 \\ 1 \end{pmatrix} \right\} \longrightarrow \text{nullity} = 2. \end{aligned}$$

Check Sample Question 10 in Math 103
(and its solution)

to see how to compute the basis

of ~~range~~, column space, row space, and null space: