Sample Solutions for Sample Questions 15

1. $$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

Compute $$A \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 4 \\ 5 \end{pmatrix}$$

So $$f(\vec{v}) = 1E_{12} + 2E_{13} + 4E_{22} + 5E_{23}$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 4 & 5 \end{pmatrix}.$$

2. No such matrix exist.

$\cancel{\mathrm{Rep}_B(\vec{v})}$ $$\mathrm{Rep}_B\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

To find $f(\vec{v}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$,

need to find $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ such that $\mathrm{Rep}_B(\vec{v}) = \vec{x}$.

But $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ has no solution.

So such $\vec{v}$ does not exist.

3. Solve $A\vec{x} = \vec{0}$.

$$A \rightsquigarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & \end{pmatrix} \\ \begin{array}{cccccc} & & \uparrow & & & \uparrow \end{array} \end{array}$$

free variables.

Find nullspace(A):

- set $x_3 = 1$, $x_6 = 0$ $\Rightarrow$ solve $A\vec{\beta}_1 = \vec{0}$ and get $\vec{\beta}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

- set $x_3 = 0$, $x_6 = 1$ $\Rightarrow$ solve $A\vec{\beta}_2 = \vec{0}$ and get $\vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \text{nullspace}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Therefore,

$$\text{nullspace}(f) = \text{span}\left\{ \underset{\text{Rep}_B}{\uparrow}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \underset{\text{Rep}_B}{\uparrow}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \Rightarrow \text{nullity} = 2$$

$$= \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

4. range(f) corresponds to the column space of A.

$$A \rightsquigarrow \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & \\ \end{pmatrix} \\ \begin{array}{cccccc} \uparrow & \uparrow & & \uparrow & \uparrow & \\ x_1 & x_2 & & x_4 & x_5 & \end{array} \end{array}$$

x_1 x_2 x_4 x_5 leading variables.

So

$$\text{Colspace}(A) = \{\text{1st, 2nd, 4th, 5th columns of } A\}.$$

$$= \text{Span}^{*}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\uparrow$ Rep_B

Therefore,

$$\text{range}(f) = \text{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \Rightarrow \text{rank} = 4.$$

$$= \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

That's also why $f(\vec{v}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ has no solution, since $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is not in range(f).

5. ① Check.

$$(f+g)(p(x)) = f(p(x)) + g(p(x))$$

$$= (x+1)\cdot p(x) + (x-1)\, p(x)$$

$$= 2x \cdot p(x).$$

②. Find $A = \begin{pmatrix} | & | \\ Rep_D(f(1)) & Rep_D(f(x)) \\ | & | \end{pmatrix}$

$$= \begin{pmatrix} | & | \\ Rep_D(x+1) & Rep_D(x^2+x) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

③. Find $B = \begin{pmatrix} | & | \\ Rep_D(g(1)) & Rep_D(g(x)) \\ | & | \end{pmatrix}$

$$= \begin{pmatrix} | & | \\ Rep_D(x-1) & Rep_D(x^2-x) \\ | & | \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

④ Find $C = \begin{pmatrix} | & | \\ Rep_D((f+g)(1)) & Rep_D((f+g)(x)) \\ | & | \end{pmatrix}$

$$= \begin{pmatrix} | & | \\ Rep_D(2x) & Rep(2x^2) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

⑤ Check $A+B=C$. True.

5. [conti.] ⑥ one-to-one ? onto?

$$\mathrm{Rep}_{B,D}(f+g) = C = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{has rank } 2.$$

rank $=$ # of columns $\Rightarrow$ one-to-one

rank $\neq$ # of ~~columns~~ rows $\Rightarrow$ not onto.

6. ① check:

$$g \circ f(p(x)) = g(f(p(x))) = g(p'(x)) = p''(x)$$

②. Find
$$A = \begin{pmatrix} \mathrm{Rep}_{B_2}(f(1)) & \mathrm{Rep}_{B_2}(f(x)) & \mathrm{Rep}_{B_2}(f(x^2)) & \mathrm{Rep}_{B_2}(f(x^3)) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Rep}_{B_2}(0) & \mathrm{Rep}_{B_2}(1) & \mathrm{Rep}_{B_2}(2x) & \mathrm{Rep}_{B_2}(3x^2) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

③ Find
$$B = \begin{pmatrix} \mathrm{Rep}_{B_1}(g(1)) & \mathrm{Rep}_{B_1}(g(x)) & \mathrm{Rep}_{B_1}(g(x^2)) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Rep}_{B_1}(0) & \mathrm{Rep}_{B_1}(1) & \mathrm{Rep}_{B_1}(2x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

④ ~~③~~ Find
$$C = \begin{pmatrix} \mathrm{Rep}_{B_1}(g(f(1))) & \mathrm{Rep}_{B_1}(g(f(x))) & \mathrm{Rep}_{B_1}(g(f(x^2))) & \mathrm{Rep}_{B_1}(g(f(x^3))) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Rep}_{B_1}(0) & \mathrm{Rep}_{B_1}(0) & \mathrm{Rep}_{B_1}(2) & \mathrm{Rep}_{B_1}(6x) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

⑤ ~~④~~ Check $BA = C$. True.

6. [Conti.]

⑥ one-to-one ? onto?

$$\mathrm{Rep}_{B_3, B_1}(g \circ f) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$ has rank 2.

rank $\neq$ # of columns $\Rightarrow$ not one-to-one

rank $=$ # of rows $\Rightarrow$ onto.

7. ~~Be~~ Let $A = \mathrm{Rep}_{B,D}(id) = \begin{pmatrix} | & | & | \\ \mathrm{Rep}_D(id(1)) & \mathrm{Rep}_D(id(x)) & \mathrm{Rep}_D(id(x^2)) \\ | & | & | \end{pmatrix}$

$$= \begin{pmatrix} | & | & | \\ \mathrm{Rep}_D(1) & \mathrm{Rep}_D(x) & \mathrm{Rep}_D(x^2) \\ | & | & | \end{pmatrix}.$$

Compute
$$1 = 1 \cdot 1 + 0 \cdot (x+1) + 0 \cdot (x+1)^2$$
$$x = -1 \cdot 1 + 1 \cdot (x+1) + 0 \cdot (x+1)^2$$
$$x^2 = 1 \cdot 1 - 2(x+1) + 1\,(x+1)^2$$

$$\Rightarrow A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

A is nonsingular $\Rightarrow$ ~~f~~ id is nonsingular.

(rank $=$ # of columns $=$ # of rows $= 3$)

(Indeed, id is of course one-to-one and onto, so it is nonsingular.)