Sample Solutions for Sample Questions 3.

1. Assume $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} = u_1(v_1 + w_1) + \dots + u_n(v_n + w_n)$$

$$= (u_1 v_1 + \dots + u_n v_n) + (u_1 w_1 + \dots + u_n w_n)$$

$$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$$

2. Claim: C-S inequality $\Longrightarrow$ Δ inequality.

Suppose $|\vec{u} \cdot \vec{v}| \le |\vec{u}||\vec{v}|$ for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$\cancel{\Rightarrow |\vec{u} \cdot \vec{v}|^2 \le |\vec{u}|^2 |\vec{v}|^2}$

$\cancel{\Rightarrow |\vec{u}|^2 +}$

Now check.

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$$

$$\le |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}| = (|\vec{u}| + |\vec{v}|)^2.$$

$$\Rightarrow |\vec{u} + \vec{v}| \le |\vec{u}| + |\vec{v}|.$$

3. Solve $a\begin{pmatrix}1\\4\end{pmatrix}+b\begin{pmatrix}1\\5\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix} \Leftrightarrow \begin{pmatrix}1&1\\4&5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$.

$$\left(\begin{array}{cc|c}1&1&2\\4&5&3\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&1&2\\0&1&-5\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&0&7\\0&1&-5\end{array}\right).$$

$\Rightarrow$ unique solution $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}7\\-5\end{pmatrix}$

4. Solve $a\begin{pmatrix}2\\1\\0\\1\end{pmatrix}+b\begin{pmatrix}3\\0\\0\\2\end{pmatrix}=\begin{pmatrix}1\\0\\1\\1\end{pmatrix} \Leftrightarrow \begin{pmatrix}2&3\\1&0\\0&0\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\0\\\cancel{4}\,1\\1\end{pmatrix}$

$$\Rightarrow \left(\begin{array}{cc|c}2&3&1\\1&0&0\\0&0&1\\1&2&1\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&0&0\\2&3&1\\0&0&1\\1&2&1\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&0&0\\0&3&1\\0&0&1\\0&2&1\end{array}\right)$$

其實不用消 ~~這~~ 這裡就看得出来無解

$$\rightsquigarrow \left(\begin{array}{cc|c}1&0&0\\0&3&1\\0&0&1\\0&0&\frac{1}{3}\end{array}\right) \begin{array}{l}\\ \\ \leftarrow \text{zero} = \text{nonzero}\\ \leftarrow \text{zero} = \text{non zero.}\end{array}$$

$\Rightarrow$ no Solution.

$\vec{v}$ cannot be written as a linear combination of vectors in X.

5.

augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & -7 & 2 & -11 \\ 0 & 1 & 2 & 5 \end{array}\right)$$

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ & 1 & 2 & 5 \\ & -7 & 2 & -11 \end{array}\right) \xrightarrow[7\rho_2+\rho_3]{-3\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ & 1 & 2 & 5 \\ & 0 & 16 & 24 \end{array}\right)$$

$$\xrightarrow{\frac{1}{16}\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ & 1 & 2 & 5 \\ & & 1 & \frac{3}{2} \end{array}\right) \xrightarrow[-2\rho_3+\rho_2]{7\rho_3+\rho_1} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ & 1 & 0 & 2 \\ & & 1 & \frac{3}{2} \end{array}\right)}_{\text{no free variable.}}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 2 \\ 3/2 \end{pmatrix}$$

$$\text{General solution} = \left\{ \begin{pmatrix} 1/2 \\ 2 \\ 3/2 \end{pmatrix} \right\}.$$

a set of the unique solution

6.

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -1 & -1 & 1 \\ 3 & 0 & -2 & 4 \end{array}\right) \underset{-3\rho_1+\rho_3}{\overset{-2\rho_1+\rho_2}{\rightsquigarrow}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -5 \end{array}\right)$$

$$\overset{-\rho_2+\rho_3}{\rightsquigarrow} \underbrace{\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{echelon form, but not reduced.}} \overset{-\frac{1}{3}\rho_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ & 1 & -\frac{1}{3} & \frac{5}{3} \\ & & & \end{array}\right)$$

$$\overset{-\rho_2+\rho_1}{\rightsquigarrow} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & \frac{4}{3} \\ & 1 & -\frac{1}{3} & \frac{5}{3} \\ & & & \end{array}\right)}_{\text{reduced echelon form.}}$$

free.

Find $\vec{p}$ by solving $R\vec{v} = \vec{r}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & \frac{4}{3} \\ & 1 & -\frac{1}{3} & \frac{5}{3} \end{array}\right) \Rightarrow \vec{p} = \begin{pmatrix} 4/3 \\ 5/3 \\ 0 \end{pmatrix}$$

0

Find $\vec{\beta}_1$ by solving $R\vec{v} = \vec{0}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0\,\cancel{\frac{4}{3}} \\ & 1 & -\frac{1}{3} & 0\,\cancel{\frac{5}{3}} \end{array}\right) \longrightarrow \vec{\beta}_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

1

General solution

$$= \{\vec{p} + c_1\vec{\beta}_1 : c_1 \in \mathbb{R}\}.$$

7.

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 2 & 1 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2 \ \cancel{2\rho_1}} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & -1 & 1 \\ 0 & -3 & -3 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ & -3 & -3 & -1 & 1 \\ & & & & \end{array}\right) \xrightarrow{-\frac{1}{3}\rho_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \\ & & & & \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_1} \begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \\ & & & & \end{array}\right) \\ \begin{array}{cccc} & & \uparrow & \uparrow \\ & & \text{free} & \text{free} \end{array} \end{array}$$

~~So~~ Find $\vec{p}$ by solving $R\vec{v} = \vec{r}$.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right) \\ \begin{array}{cccc} & & \uparrow & \uparrow \\ & & 0 & 0 \end{array} \end{array} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \end{pmatrix} = \vec{p}$$

Find $\vec{\beta}_1, \vec{\beta}_2$ by solving $R\vec{v} = \vec{0}$.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & 0 \\ & 1 & 1 & \frac{1}{3} & 0 \end{array}\right) \\ \begin{array}{cccc} & & \uparrow & \uparrow \\ & & 1 & 0 \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \vec{\beta}_1$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & 0 \\ & 1 & 1 & \frac{1}{3} & 0 \end{array}\right) \\ \begin{array}{cccc} & & \uparrow & \uparrow \\ & & 0 & 1 \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} = \vec{\beta}_2$$

General solution $= \{\vec{p} + c_1\vec{\beta}_1 + c_2\vec{\beta}_2 \mid c_1, c_2 \in \mathbb{R}\}$.