Sample Solutions for Sample Questions 4.

1. ~~one lea~~ no leading: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

one leading: $\begin{pmatrix} 1 & ? & ? \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & ? \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

two leading: $\begin{pmatrix} 1 & 0 & ? \\ 0 & 1 & ? \end{pmatrix}$, $\begin{pmatrix} 1 & \cancel{0}\,? & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Each ? can be 0 or 1.

$$\Rightarrow \quad 1 + (4+2+1) + \underset{(4+2+1)}{\cancel{(4+1)}} = \underset{15}{\cancel{13}}.$$

2. For example, the reduced echelon form can be

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \cancel{0}\,1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The idea is to use row operations to construct a matrix A with all entries nonzero.

$$R \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{A}.$$

3. We need to show $\sim$ is symmetric, transitive, and reflexive.

- (symmetric):

  Suppose $a \sim b$.

  integer

  $\Rightarrow a-b = 3k, \quad k \in \mathbb{Z}$

  $\Rightarrow b-a = 3(-k), \quad -k \in \mathbb{Z}$

  $\Rightarrow b \sim a$.

- (transitive).

  Suppose $a \sim b$ and $b \sim c$.

  $\Rightarrow a-b = 3k, \quad b-c = 3h, \quad k, h \in \mathbb{Z}$.

  $\Rightarrow a-c = 3k+3h = 3(k+h), \quad k+h \in \mathbb{Z}$.

  $\Rightarrow a \sim c$.

- (reflexive).

  Since $a-a = 0 = 3 \cdot 0, \quad 0 \in \mathbb{Z}$,

  $a \sim a$.

4.

$$\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ -8 & -10 \end{pmatrix}$$

$$\Rightarrow (3 \;\; -5) \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} = (11 \;\; 14)$$

$$\Rightarrow 3 \cdot (2 \;\; 3) + (-5)(-1 \;\; -1) = (11 \;\; 14)$$

5.

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \xrightarrow{2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{-\rho_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{-\rho_2} \underbrace{\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}}_{2\rho_2+\rho_1} \underbrace{\begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}}_{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

You may double check if this is true.

$$\Rightarrow \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\Rightarrow 2\begin{pmatrix} 1 & 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

6.

$$\begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} \xrightarrow[-8\rho_1+\rho_2]{-5\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[-\rho_2+\rho_3]{-2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\underbrace{\begin{pmatrix} 1 & -2 & \\ & 1 & \\ & -1 & 1 \end{pmatrix}}_{\substack{-2\rho_2+\rho_1 \\ -\rho_2+\rho_3}} \underbrace{\begin{pmatrix} 1 & & \\ -5 & 1 & \\ -8 & & 1 \end{pmatrix}}_{\substack{-5\rho_1+\rho_2 \\ -8\rho_1+\rho_2}} \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 11 & -2 & 0 \\ -5 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow (-5)\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} + 1\begin{pmatrix} 5 & 11 & 2 \end{pmatrix} + 0\begin{pmatrix} 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$$

7. Suppose A is $m \times n$.

Let a_{ij} be the ij-entry of A.

Fix a number k.

Let $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ be the k-th column in C and B, respectively.

$$m\left(\overset{n}{A} \right)\left({}_{n}\big[\, C \right) = \left({}_{n}\big[\, B \right)$$

Then

$$c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} = b_1$$

$$c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n} = b_2$$

$$\vdots$$

$$c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn} = b_m$$

So

$$c_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

~~Since~~ Columns of A. k-th column of B.

Since k can be chosen arbitrarily, each row of B is a linear combination of columns of A.