Sample Solutions for Sample Questions 5

1. Let $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

Then the inner product of $\vec{u}, \vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n .$$

2. singular : $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

nonsingular : $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

3. $|\vec{u}| = \sqrt{1^2 + 1^2 + 0^2 + 0^2} = \sqrt{2}$

$|\vec{v}| = \sqrt{3 + 3 + 1^2 + 1^2} = \sqrt{8}$

$\vec{u} \cdot \vec{v} = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

$$\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{2\sqrt{3}}{\sqrt{16}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ (or } 30^\circ)$$

4. If $a = 0$

$$A \rightsquigarrow \begin{pmatrix} c & d \\ 0 & b \end{pmatrix},$$

then A is nonsingular $\Leftrightarrow$ $\cancel{cd \neq 0}$ $cb \neq 0$

$\Leftrightarrow$ $\det(A) = ad - bc \neq 0.$

If $a \neq 0$

$$A \rightsquigarrow \begin{pmatrix} a & b \\ 0 & d - b \cdot \frac{c}{a} \end{pmatrix}$$

then A is nonsingular $\Leftrightarrow$ $a\left(d - b \cdot \frac{c}{a}\right) \neq 0$

$\Leftrightarrow$ $\underbrace{ad - bc}_{\substack{\| \\ \det(A)}} \neq 0.$

5. Suppose $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

Recall that $\det(A) = aei + bfg + cdh - ceg - bdi - afh$,

Observation 1. If $a = d = g = 0 \Rightarrow \det(A) = 0$.

Observation 2. Let A' be obtained from A by swapping two rows. Then $\cancel{\det(A) = 0 \Leftrightarrow}$ $\det(A') = -\det(A)$.

(so $\det(A) \neq 0 \Leftrightarrow \det(A') \neq 0$.)

Observation 3. Let A' be obtained from A by swapping two rows.

Then A is nonsingular $\Leftrightarrow$ A' is nonsingular.

[5. continued].

Let A be a 3×3 matrix.

~~Case 1~~ Claim: A is nonsingular $\Leftrightarrow$ $\det(A) \neq 0$

Case 1: Suppose $a = d = g = 0$.

Then A is singular and $\det(A) = 0$.

So the claim is true.

Case 2: Suppose one of a, d, g is nonzero.

By Observations 2, 3,

we may assume $a \neq 0$.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b & c \\ 0 & e-\frac{d}{a}b & f-\frac{d}{a}c \\ 0 & h-\frac{g}{a}b & i-\frac{g}{a}c \end{pmatrix}$$

A is nonsingular $\Leftrightarrow$ $\begin{pmatrix} e-\frac{d}{a}b & f-\frac{d}{a}c \\ h-\frac{g}{a}b & i-\frac{g}{a}c \end{pmatrix}$ is nonsingular

$$\Leftrightarrow a\cdot\left[\left(e-\frac{d}{a}b\right)\left(i-\frac{g}{a}c\right)-\left(f-\frac{d}{a}c\right)\left(h-\frac{g}{a}b\right)\right] \neq 0.$$

$$\Leftrightarrow aei - dbi - egc + \frac{dgcb}{a} - afh + dch + fgb - \frac{dcgb}{a} \neq 0$$

$$\Leftrightarrow \det(A) \neq 0.$$

6. I do not know the answer.

But I can do $k=7$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & ? & ? \end{pmatrix}.$$

~~You~~ Please let me know if you can do $k<7$.

7. $j=1$. Solve $A\vec{v}_1 = \vec{e}_1$.

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 2 & -4 & 2 & 0 \\ 2 & 3 & -6 & 5 & 0 \\ 3 & 3 & -9 & 4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & -3 & 0 & \cancel{2}4 \\ & 1 & -1 & 0 & 2 \\ & & 1 & 0 & 5 \\ & & & 1 & -3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \overset{19}{\cancel{17}} \\ & 1 & 0 & 0 & 7 \\ & & 1 & 0 & 5 \\ & & & 1 & -3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & & & & \overset{12}{\cancel{16}} \\ & 1 & & & 7 \\ & & 1 & & 5 \\ & & & 1 & -3 \end{array}\right)$$

When you try to solve $A\vec{v}_2 = \vec{e}_2$,
you will notice the left ~~had~~ hand side
is doing the same process;
only the right hand side is different.

[7. continued]

So solve everything together.

$$\left(A \;\middle|\; \begin{matrix} | & | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ | & | & | & | \end{matrix} \right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 2 & 3 & -6 & 5 & 0 & 0 & 1 & 0 \\ 3 & 3 & -9 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 1 & 1 & 0 & 0 & 0 \\ & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ & & 1 & 2 & -1 & -1 & 1 & 0 \\ & & & 1 & -3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 1 & -3 & 0 & 4 & 0 & 0 & -1 \\ & 1 & -1 & 0 & 2 & 1 & 0 & -1 \\ & & 1 & 0 & 5 & -1 & 1 & -2 \\ & & & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 19 & -3 & 3 & -7 \\ & 1 & 0 & 0 & 7 & 0 & 1 & -3 \\ & & 1 & 0 & 5 & -1 & 1 & -2 \\ & & & 1 & -3 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|cccc} 1 & & & & 12 & -3 & 2 & -4 \\ & 1 & & & 7 & 0 & 1 & -3 \\ & & 1 & & 5 & -1 & 1 & -2 \\ & & & 1 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{matrix}$$