Sample Questions 6

1. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & -4 & 2 \\ 2 & 3 & -6 & 5 \\ 3 & 3 & -9 & 4 \end{bmatrix}.$$
Find the matrix B such that $\left[\ \mathbf{I}_4 \mid \mathbf{B}\ \right]$ is the reduced echelon form of $\left[\ \mathbf{A} \mid \mathbf{I}_4\ \right]$. Also, verify that $\mathbf{BA} = \mathbf{AB} = \mathbf{I}_4$

2. Name the zero vector for each of these vector spaces.
   (a) The space of polynomials of degree $\leqslant 3$.
   (b) The space of 2×4 matrices.
   (c) The space of continuous real-valued functions on the closed interval $[0, 1]$.
   (d) The space of real-valued functions on the natural numbers.

3. In the given vector space, find the additive inverse of the vector.
   (a) Space: polynomials of degree $\leqslant 3$; vector: $-3 - 2x + x^2$.
   (b) Space: 2×2 matrices; vector: $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$.
   (c) Space: $\{ae^x + be^{-x} \mid a, b \in \mathbb{R}\}$; vector: $3e^x - 2e^{-x}$.

4. Given an $m \times n$ matrix $\mathbf{A}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, show that
$$V = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b}\}$$
is a vector space if and only if $\mathbf{b} = \mathbf{0}$.

5. Let $M_{n\times n}$ be the family of all $n \times n$ matrices and $\mathbf{O}$ the zero matrix. For a fixed $\mathbf{A} \in M_{n\times n}$, show that
$$V = \{\mathbf{X} \in M_{n\times n} \mid \mathbf{AX} = \mathbf{O}\}$$
is a vector space.

6. Show that each of these is not a vector space.
   (a) $\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x + y + z = 1\}$
   (b) $\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
   (c) $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$

7. Show that the set $\mathbb{R}^+$ of positive real numbers with the two operations $\oplus$ and $\otimes$ is a vector space when we define $x \oplus y = x \cdot y$ and $r \otimes x = x^r$. Here $+$ is the usual addition and x^r means the r-th power of x under the usual multiplication.